Undoing Trig Substitution

Professor Miller plays a game in which students give him a trig function and an inverse trig function, and then he tries to compute their composition. As we've seen, this is sometimes the final step in integration by trig substitution.

$$\_\_(\text{arc}\_\_x) = ?$$

Example: $\tan(\operatorname{arccsc} x) = ?$

Question: Isn't $\tan(\operatorname{arccsc} x)$ acceptable as a final answer?

Answer: What does "acceptable" mean? The expression $-\csc(\arctan x)$ was a *correct* final answer, but $\dfrac{\sqrt{1+x^2}}{x}$ is a nicer, more insightful, and probably more useful answer.

To simplify $\tan(\operatorname{arccsc} x)$ we draw a triangle illustrating an angle whose cosecant is x; see Figure 1. We know that

$$x = \csc\theta = \frac{1}{\sin\theta} = \frac{\text{hyp}}{\text{opp}}$$

so we choose convenient values x and 1 to be the lengths of the hypotenuse and opposite side.

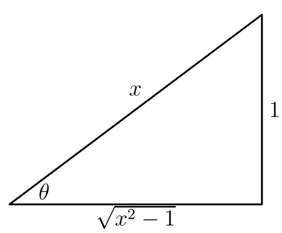

Figure 1: $\theta = \operatorname{arccsc} x$ so $x = \csc\theta$.

Once we've drawn our triangle we can compute that the length of the adjacent side must be $\sqrt{x^2-1}$, and so

$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{1}{\sqrt{x^2-1}}.$$

Since $x = \csc\theta$, we have:

$$\tan(\operatorname{arccsc} x) = \tan\theta = \frac{1}{\sqrt{x^2-1}}.$$

Whenever you have to undo a trig substitution, this technique is likely to be useful.