Example of Trig Substitution: $\int \frac{dx}{x^2\sqrt{1+x^2}}$

$$\int \frac{dx}{x^2\sqrt{1+x^2}} = ?$$

This is an ugly integral. The square root is the ugliest part, so we'll try to rewrite it in such a way that we can get rid of the square. If we let $x = \tan\theta$ then the identity $\sec^2\theta = 1 + \tan^2\theta$ will allow this. We'll then have $dx = \sec^2\theta\, d\theta$:

$$\begin{aligned}\int \frac{dx}{x^2\sqrt{1+x^2}} &= \int \frac{\sec^2\theta\, d\theta}{\tan^2\theta\sqrt{1+\tan^2\theta}} \\ &= \int \frac{\sec^2\theta\, d\theta}{\tan^2\theta\sqrt{\sec^2\theta}} \\ &= \int \frac{\sec^2\theta\, d\theta}{\tan^2\theta\sec\theta} \\ &= \int \frac{\sec\theta\, d\theta}{\tan^2\theta}\end{aligned}$$

When faced with an assortment of different trig functions like this one, it's a good idea to rewrite everything in terms of $\sin\theta$ and $\cos\theta$:

$$\begin{aligned}\int \frac{dx}{x^2\sqrt{1+x^2}} &= \int \frac{\frac{1}{\cos\theta}\, d\theta}{\frac{\sin^2\theta}{\cos^2\theta}} \\ &= \int \frac{\cos^2\theta\, d\theta}{\cos\theta\sin^2\theta} \\ &= \int \frac{\cos\theta\, d\theta}{\sin^2\theta}\end{aligned}$$

The ugliest part of this integral is the $\sin^2\theta$ in the denominator. Since $\cos\theta\, d\theta$ is the derivative of $\sin\theta$, we make the substitution $u = \sin\theta$, $du = \cos\theta\, d\theta$:

$$\begin{aligned}\int \frac{dx}{x^2\sqrt{1+x^2}} &= \int \frac{\cos\theta\, d\theta}{\sin^2\theta} \\ &= \int \frac{du}{u^2} \\ &= -\frac{1}{u} + c\end{aligned}$$

Now we have to reverse our substitutions:

$$\begin{aligned}\int \frac{dx}{x^2\sqrt{1+x^2}} &= -\frac{1}{\sin\theta} + c \\ &= -\csc\theta + c\end{aligned}$$

It's not clear how to undo the substitution $x = \tan\theta$. Luckily there is a general method for undoing substitutions like this, which is to go back to thinking of trig functions as ratios of side lengths of a right triangle.

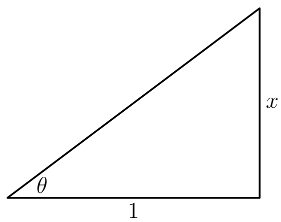

Figure 1: Undoing trig substitution.

We know $x = \tan\theta$ and we know that $\tan\theta$ equals the length of the leg opposite θ divided by the length of the leg adjacent to θ. Figure 1 shows a right triangle with an angle θ, an opposite leg of length x, and an adjacent leg of length 1.

The Pythagorean theorem tells us that the hypotenuse must have length $\sqrt{1+x^2}$. Now we can deduce that:

$$\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{\sqrt{1+x^2}}{x}.$$

Hence,

$$\int \frac{dx}{x^2\sqrt{1+x^2}} \quad = \quad -\frac{\sqrt{1+x^2}}{x} + c.$$

In the process of computing this integral we saw the following: trig substitution, rewriting trig functions in terms of sine and cosine, direct substitution, and undoing trig substitution.

What actually happened when we undid that trig substitution was that we computed $\csc(\arctan(x))$. In other words, we composed a trig function with the inverse of another trig function.