

## Area of Part of a Circle

Given a circle of radius  $a$ , cut out a tab of height  $b$ . What is the area of this tab? (See Figure 1.)

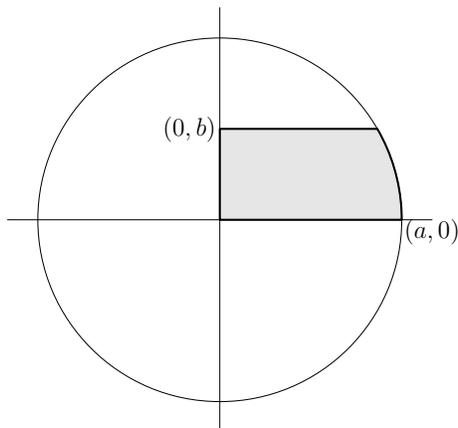


Figure 1: Tab cut out of a circle.

One way to compute the area would be split the area into vertical strips and integrate with respect to  $x$ :

$$\text{Area} = \int y \, dx.$$

This is awkward, because near the end the height of the region changes from a constant  $y = b$  to the height of the circle  $y = \sqrt{a^2 - x^2}$ .

What if we integrate with respect to  $y$ ? That seems to work better; there is a single simple expression for the length of each horizontal strip:  $x = \sqrt{a^2 - y^2}$ .

$$\begin{aligned} \text{Area} &= \int_0^b x \, dy \\ &= \int_0^b \sqrt{a^2 - y^2} \, dy \end{aligned}$$

We don't yet have a rule for integrating functions of this form. Considering that this integral arose from a question about a circle, it's not surprising that trigonometry will play a role in its solution.

When working with circles it often helps to use polar coordinates. In this case, note that the upper right hand corner of the region has polar coordinates  $(a \cos \theta_0, a \sin \theta_0)$  where  $\theta_0$  is the angle shown in Figure 2.

In general,  $x = a \cos \theta$  and  $y = a \sin \theta$ . If we substitute  $y = a \sin \theta$  into our integrand we get:

$$x = \sqrt{a^2 - y^2}$$

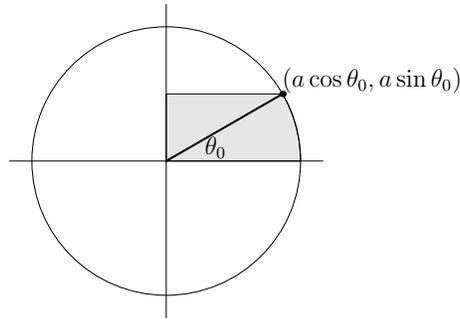


Figure 2: Polar coordinates of a point.

$$\begin{aligned}
 &= \sqrt{a^2 - a^2 \sin^2 \theta} \\
 &= a\sqrt{1 - \sin^2 \theta} \\
 &= a\sqrt{\cos^2 \theta} \\
 x &= a \cos \theta
 \end{aligned}$$

Changing to polar coordinates made our integrand look much nicer; we've gone from an integrand with a square root and no trig functions to an integrand with trig functions and no square root.

If we're going to use the substitution  $y = a \sin \theta$  in our integral, we'll also need to replace  $dy$  by something in polar coordinates.

$$\begin{aligned}
 y &= a \sin \theta \\
 dy &= a \cos \theta d\theta
 \end{aligned}$$

Plugging in, we get:

$$\begin{aligned}
 \int \sqrt{a^2 - y^2} dy &= \int (a \cos \theta)(a \cos \theta d\theta) \\
 &= a^2 \int \cos^2 \theta d\theta
 \end{aligned}$$

We computed the integral of  $\cos^2 x$  earlier in the lecture:

$$\int \cos^2 x dx = \frac{x}{2} + \frac{\sin 2x}{4} + C.$$

Plugging this in, we get:

$$\int \sqrt{a^2 - y^2} dy = a^2 \left( \frac{\theta}{2} + \frac{\sin(2\theta)}{4} \right) + c.$$

Now we'd like to rewrite our solution in terms of the original variable  $y$  so that we can plug in the limits of integration. In order to do this, it's helpful to rewrite  $\sin(2\theta)$  using the double angle formula  $\sin(2\theta) = 2 \sin \theta \cos \theta$ .

$$\begin{aligned}
\int \sqrt{a^2 - y^2} dy &= a^2 \left( \frac{\theta}{2} + \frac{\sin(2\theta)}{4} \right) + c \\
&= a^2 \left( \frac{\theta}{2} + \frac{\sin \theta \cos \theta}{2} \right) + c \\
&= \left( \frac{a^2 \theta}{2} + \frac{a \sin \theta a \cos \theta}{2} \right) + c.
\end{aligned}$$

Next we solve  $y = a \sin \theta$  for  $y$  and plug in:

$$\theta = \arcsin\left(\frac{y}{a}\right).$$

Since  $a \sin \theta = y$  and  $a \cos \theta = x = \sqrt{a^2 - y^2}$ , we get:

$$\int \sqrt{a^2 - y^2} dy = \left( \frac{a^2 \arcsin(y/a)}{2} + \frac{y\sqrt{a^2 - y^2}}{2} \right) + c.$$

We've used trigonometric substitution to find the indefinite integral of  $\sqrt{a^2 - y^2}$ . Whenever you see the square root of a quadratic in an integral you should think of trigonometry and  $\sin^2 \theta + \cos^2 \theta$ .

Our original problem asked us to compute the value of a definite integral; let's finish that.

$$\begin{aligned}
\int_0^b \sqrt{a^2 - y^2} dy &= \left( \frac{a^2 \arcsin(y/a)}{2} + \frac{y\sqrt{a^2 - y^2}}{2} \right) \Big|_0^b \\
&= \left( \frac{a^2 \arcsin(b/a)}{2} + \frac{b\sqrt{a^2 - b^2}}{2} \right) - 0 \\
&= \frac{a^2 \arcsin(b/a)}{2} + \frac{b\sqrt{a^2 - b^2}}{2}
\end{aligned}$$

Notice that  $\theta_0 = \arcsin(b/a)$ ; we could rewrite this answer as:

$$\text{Area} = \frac{a^2 \theta_0}{2} + \frac{b\sqrt{a^2 - b^2}}{2}$$

Does this make sense? The first term,  $\frac{\theta_0}{2} a^2$ , is exactly the area of the sector of the circle swept out by angle  $\theta_0$ . The second term,  $\frac{1}{2} b\sqrt{a^2 - b^2}$ , is the area of a triangle with base  $b$  and height  $\sqrt{a^2 - b^2}$ . In other words, it's the area of the shaded triangle shown in Figure 2.

Using some basic geometry, we've checked that our answer to this complicated calculus problem is correct.

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