

## 18.01 EXERCISES

### Unit 4. Applications of integration

#### 4A. Areas between curves.

**4A-1** Find the area between the following curves

- a)  $y = 2x^2$  and  $y = 3x - 1$       b)  $y = x^3$  and  $y = ax$ ; assume  $a > 0$   
c)  $y = x + 1/x$  and  $y = 5/2$ .      d)  $x = y^2 - y$  and the  $y$  axis.

**4A-2** Find the area under the curve  $y = 1 - x^2$  in two ways.

**4A-3** Find the area between the curves  $y = 4 - x^2$  and  $y = 3x$  in two ways.

**4A-4** Find the area between  $y = \sin x$  and  $y = \cos x$  from one crossing to the next.

#### 4B. Volumes by slicing; volumes of revolution

**4B-1** Find the volume of the solid of revolution generated by rotating the regions bounded by the curves given around the  $x$ -axis.

- a)  $y = 1 - x^2, y = 0$       b)  $y = a^2 - x^2, y = 0$       c)  $y = x, y = 0, x = 1$   
d)  $y = x, y = 0, x = a$       e)  $y = 2x - x^2, y = 0$       f)  $y = 2ax - x^2, y = 0$   
g)  $y = \sqrt{ax}, y = 0, x = ah$       h)  $x^2/a^2 + y^2/b^2 = 1, x = 0$

**4B-2** Find the volume of the solid of revolution generated by rotating the regions in 4B-1 around the  $y$ -axis.

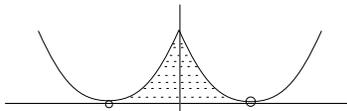
**4B-3** Show that the volume of a pyramid with a rectangular base is  $bh/3$ , where  $b$  is the area of the base and  $h$  is the height. (Show in the process that the proportions of the rectangle do not matter.)

**4B-4** Consider  $(x, y, z)$  such that  $x^2 + y^2 < 1, x > 0$  and  $0 \leq z \leq 5$ . This describes one half of cylinder (a split log). Chop out a wedge out of the log along  $z = 2x$ . Find the volume of the wedge.

**4B-5** Find the volume of the solid obtained by revolving an equilateral triangle of sidelength  $a$  around one of its sides.

**4B-6** The base of a solid is the disk  $x^2 + y^2 \leq a^2$ . Planes perpendicular to the  $xy$ -plane and perpendicular to the  $x$ -axis slice the solid in isosceles right triangles. The hypotenuse of these triangles is the segment where the plane meets the disk. What is the volume of the solid?

**4B-7** A tower is constructed with a square base and square horizontal cross-sections. Viewed from any direction perpendicular to a side, the tower has base  $y = 0$  and profile lines  $y = (x-1)^2$  and  $y = (x+1)^2$ . (See shaded region in picture.) Find the volume of the solid.



#### 4C. Volumes by shells

**4C-1** Assume that  $0 < a < b$ . Revolve the disk  $(x - b)^2 + y^2 \leq a^2$  around the  $y$  axis. This doughnut shape is known as a torus.

a) Set up the integral for volume using integration  $dx$

b) Set up the integral for volume using integration  $dy$

c) Evaluate (b).

d) (optional) Show that the (a) and (b) are the same using the substitution  $z = x - b$ .

**4C-2** Find the volume of the region  $0 \leq y \leq x^2$ ,  $x \leq 1$  revolved around the  $y$ -axis.

**4C-3** Find the volume of the region  $\sqrt{x} \leq y \leq 1$ ,  $x \geq 0$  revolved around the  $y$ -axis by both the method of shells and the method of disks and washers.

**4C-4** Set up the integrals for the volumes of the regions in 4B-1 by the method of shells. (Do not evaluate.)

**4C-5** Set up the integrals for the volumes of the regions in 4B-2 by the method of shells. (Do not evaluate.)

**4C-6** Let  $0 < a < b$ . Consider a ball of radius  $b$  and a cylinder of radius  $a$  whose axis passes through the center of the ball. Find the volume of the ball with the cylinder removed.

#### 4D. Average value

**4D-1** What is the average cross-sectional area of the solid obtained by revolving the region bounded by  $x = 2$ , the  $x$ -axis, and the curve  $y = x^2$  about the  $x$ -axis? (Cross-sections are taken perpendicular to the  $x$ -axis.)

**4D-2** Show that the average value of  $1/x$  over the interval  $[a, 2a]$  is of the form  $C/a$ , where  $C$  is a constant independent of  $a$ . (Assume  $a > 0$ .)

**4D-3** A point is moving along the  $x$ -axis, with distance function given by  $x = s(t)$ . Show that over a time interval  $[a, b]$ , the average value of its velocity  $v(t)$  is the same as its average velocity over this interval.

**4D-4** What is the average value of the square of the distance of a point  $P$  from a fixed point  $Q$  on the unit circle, where  $P$  is chosen at random on the circle? (Use coordinates; place  $Q$  on the  $x$ -axis.) Check your answer for reasonableness.

**4D-5** If the average value of  $f(t)$  between 0 and  $x$  is given by the function  $g(x)$ , express  $f(x)$  in terms of  $g(x)$ .

**4D-6** An amount of money  $A$  compounded continuously at interest rate  $r$  increases according to the law

$$A(t) = A_0 e^{rt} \quad (t = \text{time in years})$$

a) What is the average amount of money in the bank over the course of  $T$  years?

b) Suppose  $r$  and  $T$  are small. Give an approximate answer to part (a) by using the quadratic approximation to your exact answer; check it for reasonableness.

**4D-7** Find the average value of  $x^2$  in  $0 \leq x \leq b$ .

**4D-8** Find the average distance from a point on the perimeter of a square of sidelength  $a$  to the center. Find the average of the square of the distance.

**4D-9** Find the average value of  $\sin ax$  in its first hump.

#### 4D'. Work

**4D'-1** An extremely stiff spring is 12 inches long, and a force of 2,000 pounds extends it  $1/2$  inch. How many foot-pounds of work would be done in stretching it to 18 inches?

**4D'-2** A heavy metal 2 pound pail initially is filled with 10 pounds of paint. Immediately after it is filled, it is pulled up at a steady rate to the top of a building 30 feet high. While being pulled, the paint leaks out through a hole in the pail at a steady rate so that by the time it reaches the top,  $1/5$  of the paint has leaked out. How many foot-pounds of work were done pulling the pail to the top of the building?

**4D'-3** A heavy-duty rubber firehose hanging over the side of a building is 50 feet long and weighs 2 lb./foot. How much work is done winding it up on a windlass on the top of the building?

**4D'-4** Two point-particles having respective masses  $m_1$  and  $m_2$  are at  $d$  units distance. How much work is required to move them  $n$  times as far apart (i.e., to distance  $nd$ )? What is the work to move them infinitely far apart?

#### 4E. Parametric equations

**4E-1** Find the rectangular equation for  $x = t + t^2$ ,  $y = t + 2t^2$ .

**4E-2** Find the rectangular equation for  $x = t + 1/t$  and  $y = t - 1/t$  (compute  $x^2$  and  $y^2$ ).

**4E-3** Find the rectangular equation for  $x = 1 + \sin t$ ,  $y = 4 + \cos t$ .

**4E-4** Find the rectangular equation for  $x = \tan t$ ,  $y = \sec t$ .

**4E-5** Find the rectangular equation for  $x = \sin 2t$ ,  $y = \cos t$ .

**4E-6** Consider the parabola  $y = x^2$ . Find the parametrization using the slope of the curve at a point  $(x, y)$  as the parameter.

**4E-7** Find the parametrization of the circle  $x^2 + y^2 = a^2$  using the slope as the parameter. Which portion of the circle do you obtain in this way?

**4E-8** At noon, a snail starts at the center of an open clock face. It creeps at a steady rate along the hour hand, reaching the end of the hand at 1:00 PM. The hour hand is 1 meter long. Write parametric equations for the position of the snail at time  $t$ , in some reasonable  $xy$ -coordinate system.

**4E-9\*** a) What part of a train is moving backwards when the train moves forwards?

b) A circular disc has inner radius  $a$  and outer radius  $b$ . Its inner circle rolls along the positive  $x$ -axis without slipping. Find parametric equations for the motion of a point  $P$  on its outer edge, assuming  $P$  starts at  $(0, b)$ . Use  $\theta$  as parameter. (Your equations should reduce to those of the cycloid when  $a = b$ . Do they?)

c) Sketch the curve that  $P$  traces out.

d) Show from the parametric equations you found that  $P$  is moving backwards whenever it lies below the  $x$ -axis.

#### 4F. Arclength

**4F-1** Find the arclength of the following curves

$$\begin{array}{ll} \text{a) } y = 5x + 2, 0 \leq x \leq 1. & \text{b) } y = x^{3/2}, 0 \leq x \leq 1. \\ \text{c) } y = (1 - x^{2/3})^{3/2}, 0 \leq x \leq 1. & \text{d) } y = (1/3)(2 + x^2)^{3/2}, 1 \leq x \leq 2. \end{array}$$

**4F-2** Find the length of the curve  $y = (e^x + e^{-x})/2$  for  $0 \leq x \leq b$ . Hint:

$$\left(\frac{e^x - e^{-x}}{2}\right)^2 + 1 = \left(\frac{e^x + e^{-x}}{2}\right)^2$$

**4F-3** Express the length of the parabola  $y = x^2$  for  $0 \leq x \leq b$  as an integral. (Do not evaluate.)

**4F-4** Find the length of the curve  $x = t^2, y = t^3$  for  $0 \leq t \leq 2$ .

**4F-5** Find an integral for the length of the curve given parametrically in Exercise 4E-2 for  $1 \leq t \leq 2$ . Simplify the integrand as much as possible but do not evaluate.

**4F-6** a) The cycloid given parametrically by  $x = t - \sin t, y = 1 - \cos t$  describes the path of a point on a rolling wheel. If  $t$  represents time, then the wheel is rotating

at a constant speed. How fast is the point moving at each time  $t$ ? When is the forward motion ( $dx/dt$ ) largest and when is it smallest?

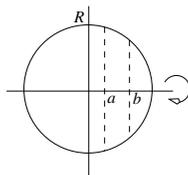
b) Find the length of the cycloid for one turn of the wheel. (Use a half angle formula.)

**4F-7** Express the length of the ellipse  $x^2/a^2 + y^2/b^2 = 1$  using the parametrization  $x = a \cos t$  and  $y = b \sin t$ . (Do not evaluate.)

**4F-8** Find the length of the curve  $x = e^t \cos t$ ,  $y = e^t \sin t$  for  $0 \leq t \leq 10$ .

### 4G. Surface Area

**4G-1** Consider the sphere of radius  $R$  formed by revolving the circle  $x^2 + y^2 = R^2$  around the  $x$ -axis. Show that for  $-R \leq a < b \leq R$ , the portion of the sphere  $a \leq x \leq b$  has surface area  $2\pi R(b - a)$ . For example, the hemisphere,  $a = 0$ ,  $b = R$  has area  $2\pi R^2$ .



**4G-2** Find the area of the segment of  $y = 1 - 2x$  in the first quadrant revolved around the  $x$ -axis.

**4G-3** Find the area of the segment of  $y = 1 - 2x$  in the first quadrant revolved around the  $y$ -axis.

**4G-4** Find an integral formula for the area of  $y = x^2$ ,  $0 \leq x \leq 4$  revolved around the  $x$ -axis. (Do not evaluate.)

**4G-5** Find the area of  $y = x^2$ ,  $0 \leq x \leq 4$  revolved around the  $y$ -axis.

**4G-6** Find the area of the astroid  $x^{2/3} + y^{2/3} = a^{2/3}$  revolved around the  $x$ -axis.

**4G-7** Consider the torus of Problem E22/1.

a) Set up the integral for surface area using integration  $dx$

b) Set up the integral for surface area using integration  $dy$

c) Evaluate (b) using the substitution  $y = a \sin \theta$ .

#### 4H. Polar coordinate graphs

**4H-1** For each of the following points given in rectangular coordinates, give its polar coordinates. (For points below the  $x$ -axis, give two expressions for its polar coordinates, using respectively positive and negative values for  $\theta$ .)

- a)  $(0, 3)$  b)  $(-2, 0)$  c)  $(1, \sqrt{3})$  d)  $(-2, 2)$   
 e)  $(1, -1)$  f)  $(0, -2)$  g)  $(\sqrt{3}, -1)$  h)  $(-2, -2)$

#### 4H-2

a) Find using two different methods the equation in polar coordinates for the circle of radius  $a$  with center at  $(a, 0)$  on the  $x$ -axis, as follows:

(i) write its equation in rectangular coordinates, and then change it to polar coordinates (substitute  $x = r \cos \theta$  and  $y = r \sin \theta$ , and then simplify).

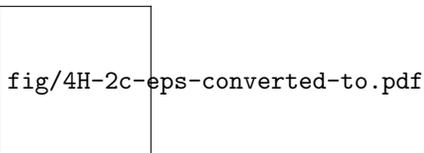
(ii) treat it as a locus problem: let  $OQ$  be the diameter lying along the  $x$ -axis, and  $P : (r, \theta)$  a point on the circle; use  $\triangle OPQ$  and trigonometry to find the relation connecting  $r$  and  $\theta$ .

b) Carry out the analogue of 4H-2a for the circle of radius  $a$  with center at  $(0, a)$  on the  $y$ -axis;  $OQ$  is now the diameter lying along the  $y$ -axis.

c) (i) Find the polar equation for the line intersecting the positive  $x$ - and  $y$ -axes respectively at  $A$  and  $B$ , and having perpendicular distance  $a$  from the origin.

(Let  $\alpha = \angle DOA$ ; use the right triangle  $DOP$  to get the equation connecting  $r, \theta, \alpha$  and  $a$ .)

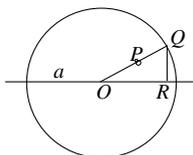
(ii) Convert your polar equation to the usual rectangular equation involving  $A$  and  $B$ , by using trigonometry.



d) In the accompanying figure, the point  $Q$  moves around the circle of radius  $a$  centered at the origin;  $QR$  is a perpendicular to the  $x$ -axis.  $P$  is a point on ray  $OQ$

such that  $|QP| = |QR|$ :  $P$  is the point inside the circle in the first two quadrants, but outside the circle in the last two quadrants.

- (i) Sketch the locus of  $P$ ; the locus is called a *cardioid* (cf. 4H-3c).
- (ii) find the polar equation of this locus.



e) The point  $P$  moves in a locus so that the product of its distances from the two points  $Q : (-a, 0)$  and  $R : (a, 0)$  is constant. Assuming the locus of  $P$  goes through the origin, determine the value of the constant, and derive the polar equation of the locus of  $P$ .

(Work with the squares of the distances, rather than the distances themselves, and use the law of cosines; the identities  $(A + B)(A - B) = A^2 - B^2$  and  $\cos 2\theta = 2\cos^2\theta - 1$  simplify the algebra and produce a simple answer at the end. The resulting curve is a *lemniscate*, cf. 4H-3g.)

**4H-3** For each of the following,

- (i) give the corresponding equation in rectangular coordinates;
- (ii) draw the graph; indicate the direction of increasing  $\theta$ .

a)  $r = \sec \theta$       b)  $r = 2a \cos \theta$       c)  $r = (a + b \cos \theta)$  (This figure is a cardioid for  $a = b$ , a limaçon with a loop for  $0 < a < b$ , and a limaçon without a loop for  $a > b > 0$ .)

d)  $r = a/(b + c \cos \theta)$  (Assume the constants  $a$  and  $b$  are positive. This figure is an ellipse for  $b > |c| > 0$ , a circle for  $c = 0$ , a parabola for  $b = |c|$ , and a hyperbola for  $b < |c|$ .)

- e)  $r = a \sin(2\theta)$  (4-leaf rose)      f)  $r = a \cos(2\theta)$  (4-leaf rose)
- g)  $r^2 = a^2 \sin(2\theta)$  (lemniscate)      h)  $r^2 = a^2 \cos(2\theta)$  (lemniscate)
- i)  $r = e^{a\theta}$  (logarithmic spiral)

#### 4I. Area and arclength in polar coordinates

**4I-1** Find the arclength element  $ds = w(\theta)d\theta$  for the curves of 4H-3.

**4I-2** Find the area of one leaf of a three-leaf rose  $r = a \cos(3\theta)$ .

**4I-3** Find the area of the region  $0 \leq r \leq e^{3\theta}$  for  $0 \leq \theta \leq \pi$

**4I-4** Find the area of one loop of the lemniscate  $r^2 = a^2 \sin(2\theta)$

**4I-5** What is the average distance of a point on a circle of radius  $a$  from a fixed point  $Q$  on the circle? (Place the circle so  $Q$  is at the origin and use polar coordinates.)

**4I-6** What is the average distance from the  $x$ -axis of a point chosen at random on the cardioid  $r = a(1 - \cos \theta)$ , if the point is chosen

a) by letting a ray  $\theta = c$  sweep around at uniform velocity, stopping at random and taking the point where it intersects the cardioid;

b) by letting a point  $P$  travel around the cardioid at uniform velocity, stopping at random; (the answers to (a) and (b) are different...)

**4I-7** Calculate the area and arclength of a circle, parameterized by  $x = a \cos \theta, y = a \sin \theta$ .

#### 4J. Other Applications

**4J-1** Suppose it takes  $k$  units of energy to lift a cubic meter of water one meter. About how much energy  $E$  will it take to pump dry a circular hole one meter in diameter and 100 meters deep that is filled with water? (Give reasoning.)

**4J-2** The amount  $x$  (in grams) of a radioactive material declines exponentially over time (in minutes), according to the law  $x = x_0 e^{-kt}$ , where  $x_0$  is the amount initially present at time  $t = 0$ . If one gram of the material produces  $r$  units of radiation/minute, about how much radiation  $R$  is produced over one hour by  $x_0$  grams of the material? (Give reasoning.)

**4J-3** A very shallow circular reflecting pool has uniform depth  $D$ , and radius  $R$  (meters). A disinfecting chemical is released at its center, and after a few hours of symmetrical diffusion outwards, the concentration of chemical at a point  $r$  meters from the center is  $\frac{k}{1+r^2}$  g/m<sup>3</sup>.

What amount  $A$  of the chemical was released into the pool? (Give reasoning.)

**4J-4** Assume a heated outdoor pool requires  $k$  units of heat/hour for each degree F it is maintained above the external air temperature.

If the external temperature  $T$  varies between  $50^\circ$  and  $70^\circ$  over a 24 hour period starting at midnight, according to  $T = 10(6 - \cos(\pi t/12))$ , how many heat units will be required to maintain the pool at a steady  $75^\circ$  temperature? (Give reasoning.)

**4J-5** A manufacturer's cost for storing one unit of inventory is  $c$  dollars/day for space and insurance. Over the course of 30 days, production  $P$  rises from 10 to 40 units/day according to  $P = 10 + t$ . Assuming no units are sold, what is the inventory cost for this period? (Give reasoning.)

**4J-6** A water tank for a town has the shape of a sphere of radius  $r$  feet, and its center is at a height  $h$  above the ground. If the weight of a cubic foot of water is  $w$  lbs., how much work is required to fill the tank when empty by pumping water from the ground? (Give reasoning using infinitesimals.)

**4J-6** Divide the water in the tank into thin horizontal slices of width  $dy$ .

If the slice is at height  $y$  above the center of the tank, its radius is  $\sqrt{r^2 - y^2}$ .

volume of water in the slice =  $\pi(r^2 - y^2) dy$

weight of water in the slice =  $\pi w(r^2 - y^2) dy$

work to lift this slice from the ground =  $\pi w(r^2 - y^2) dy (h + y)$ .

$$\text{Total work} = \int_{-r}^r \pi w(r^2 - y^2)(h + y) dy = \pi w \int_{-r}^r (r^2 h + r^2 y - h y^2 - y^3) dy = \pi w \left[ r^2 h y + \frac{r^2 y^2}{2} - h \frac{y^3}{3} - \frac{y^4}{4} \right]_{-r}^r.$$

The even powers of  $y$  have the same value at  $-r$  and  $r$ , so contribute 0 to the value; we get

$$= \pi w h \left[ r^2 y - \frac{y^3}{3} \right]_{-r}^r = 2\pi w h \left( r^3 - \frac{r^3}{3} \right) = \frac{4}{3} \pi w h r^3.$$

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