

Estimating $\ln(5)$

- a) Use the mean value theorem and the fundamental theorem of calculus to find upper and lower bounds on $\int_1^5 \frac{1}{x} dx$.
- b) Compute $\int_1^5 \frac{1}{x} dx$.
- c) Does your answer to (a) provide a good estimate of the value of $\ln(5)$?

Solution

- a) Use the mean value theorem and the fundamental theorem of calculus to find upper and lower bounds on $\int_1^5 \frac{1}{x} dx$.

The mean value theorem tells us that if F is differentiable on $a < x < b$ and continuous on $a \leq x \leq b$, then there exists some value c with $a < c < b$ such that:

$$\frac{F(b) - F(a)}{b - a} = F'(c).$$

In our example, $F(b) - F(a) = \int_1^5 \frac{1}{x} dx$ and $F'(x) = \frac{1}{x}$. We're concerned with the interval $[1, 5]$, so the mean value theorem says that:

$$\frac{\int_1^5 \frac{1}{x} dx}{5 - 1} = \frac{1}{c}$$

for some c between 1 and 5. In other words, $\int_1^5 \frac{1}{x} dx = \frac{4}{c}$ for some c in $(1, 5)$.

On the interval between 1 and 5, $F'(x) = \frac{1}{x}$ is at its greatest when $x = 1$ and is smallest when $x = 5$. The function $F'(x)$ decreases along the entire interval from 1 to 5, so we conclude that for any c with $1 < c < 5$, $1 > \frac{1}{c} > \frac{1}{5}$.

From this we get $4 > \int_1^5 \frac{1}{x} dx > \frac{4}{5}$, or equivalently,

$$0.8 < \int_1^5 \frac{1}{x} dx < 4.$$

- b) Compute $\int_1^5 \frac{1}{x} dx$.

$$\begin{aligned} \int_1^5 \frac{1}{x} dx &= \ln(x) \Big|_1^5 \\ &= \ln(5) - \ln(1) \\ &= \ln(5) \approx 1.609 \end{aligned}$$

We learned in part (a) that our answer should be between 0.8 and 4, so this is probably correct. We could also check our work by looking at a graph of the function $F'(x) = \frac{1}{x}$.

c) Does your answer to (a) provide a good estimate of the value of $\ln(5)$?

The lower bound $0.8 < \ln(5)$ is somewhat helpful, but the information that $\ln(5) < 4$ is not particularly useful. Using a calculator to apply the function e^x to both sides of this inequality we learn that $5 < e^4 \approx 55$.

The mean value theorem gives us upper and lower bounds, but those may not be useful for estimation. This is just one of many methods of finding upper and lower bounds.

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