The Mean Value Theorem and Estimation

The following problem appeared on the second exam:

Given that $F'(x) = \frac{1}{1+x}$ and $F(0) = 1$, the mean value theorem implies that $A < F(4) < B$ for which A and B?

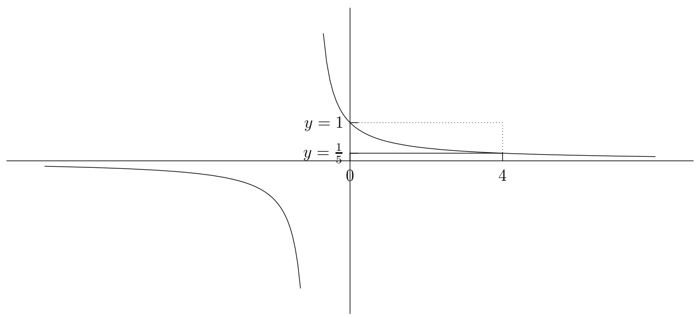

Figure 1: Graph of $F'(x) = \frac{1}{1+x}$.

To solve this, we first apply the mean value theorem in such a way that the value $F(4)$ appears, then use our knowledge of the formula for $F'(c)$ to find limits on that value. Remember that c is an unknown value between (in this case) 0 and 4.

$$\begin{aligned} F(4) - F(0) &= F'(c)(4-0) \quad \text{(Use the MVT on } F(4)\text{)} \\ &= \frac{1}{1+c} \cdot 4 \end{aligned}$$

We don't know what $\frac{1}{1+c}$ is, but we know that $\frac{1}{x}$ decreases from 0 to infinity, so:

$$1 = \frac{1}{1} > \frac{1}{1+c} > \frac{1}{1+4} = \frac{1}{5}.$$

Hence:

$$4 > \frac{1}{1+c} \cdot 4 > \frac{4}{5}.$$

We conclude that:

$$4 > F(4) - F(0) > \frac{4}{5}$$

and since $F(0) = 1$ we have:

$$5 > F(4) > \frac{9}{5}.$$

Our final answer is $A = \frac{9}{5}$ and $B = 5$.

Now let's compare this to what we can do with the fundamental theorem of calculus:

$$F(4) - F(0) = \int_0^4 \frac{dx}{1+x}$$

Based on what we know about the graph of $y = \frac{1}{x}$ and the area under it, we can deduce that:

$$F(4) - F(0) = \int_0^4 \frac{dx}{1+x} < \int_0^4 1dx = 4$$

and that

$$F(4) - F(0) = \int_0^4 \frac{dx}{1+x} > \int_0^4 \frac{1}{5}dx = \frac{4}{5}.$$

So once again we have:

$$\frac{4}{5} < F(4) - F(0) < 4.$$

Geometrically, we interpret $\int_0^4 \frac{dx}{1+x}$ as the area under a curve. We got an upper bound on the area by comparing it to the area of a rectangle whose height was the maximum value of $\frac{1}{1+x}$ on the interval, and got a lower bound by comparing to a rectangle whose hight was the minimum of $\frac{1}{1+x}$ on $[0, 4]$.

We could think of this as estimating $\int_0^4 \frac{dx}{1+x}$ by comparing it to two different Riemann sums, each with only *one* rectangle.

$$\text{lower Riemann sum} < \int_0^4 \frac{dx}{1+x} < \text{upper Riemann sum}$$