Antiderivative of x^a

What function has the derivative x^a? We know that the exponent decreases by one when we differentiate, so we guess x^{1+1}. This doesn't quite work:

$$d(x^{a+1}) = (a+1)x^a dx.$$

We have to divide both sides by the constant $(a+1)$ to get the correct answer.

$$\begin{aligned} d(\frac{x^{a+1}}{a+1}) &= x^a dx \\ \frac{x^{a+1}}{a+1} + c &= \int x^a dx \end{aligned}$$

But wait! Although it's true that $d(x^{a+1}) = (a+1)x^a dx$, it is not always true that $\int x^a dx = \frac{x^{a+1}}{a+1} + c$. When $a = -1$ the denominator is zero. However, we can still say that $\int x^a dx = \frac{x^{a+1}}{a+1} + c$ for $a \neq -1$.

What happens when $a = -1$? What is $\int \frac{1}{x} dx$?

So far we've used the formulas $\frac{d}{dx} \cos x = -\sin x$ and $\frac{d}{dx} x^{n+1} = (n+1)x^n$. An important part of integration is remembering formulas for derivatives and "reading them backward". In this case, the formula we need is $\frac{d}{dx} \ln x = \frac{1}{x}$. Using this, we get $\int \frac{1}{x} dx = \ln x + c$.

This formula is fine when $x > 0$, but $\ln x$ is not defined when x is negative. The more standard form of this equation is:

$$\int \frac{1}{x} dx = \ln |x| + c.$$

The absolute value doesn't change anything when $x \geq 0$, so we only need to check this formula when x is negative. In order to do so, we have to differentiate $\ln |x|$.

$$\begin{aligned} \frac{d}{dx} \ln |x| &= \frac{d}{dx} \ln(-x) \quad (|x| = -x \text{ when } x < 0) \\ &= \frac{1}{-x} \frac{d}{dx}(-x) \quad (\text{by the chain rule}) \\ &= -\frac{1}{-x} \\ &= \frac{1}{x} \end{aligned}$$

If we graph $\ln |x|$ we can see that this function does have slope $\frac{1}{x}$.

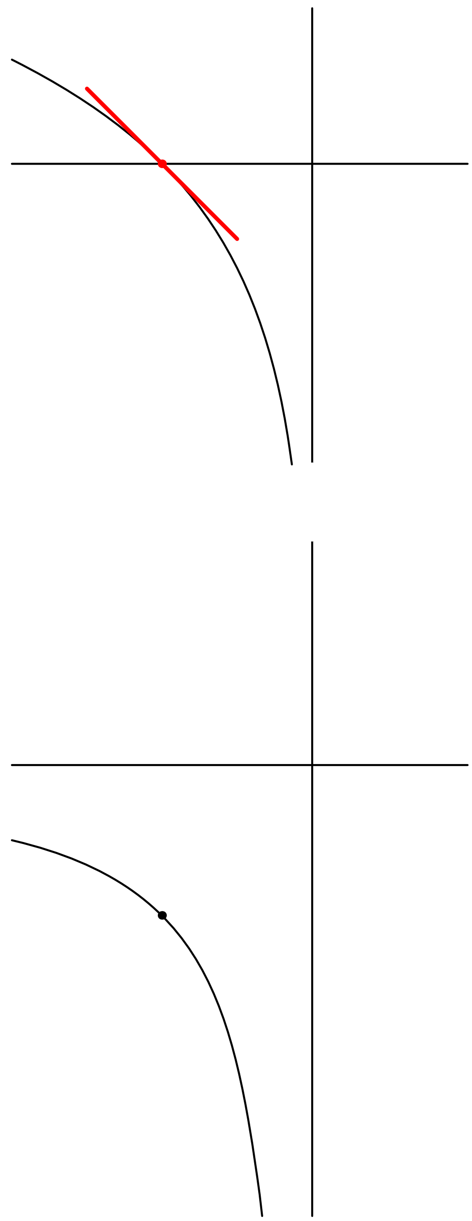

Figure 1: Graphs of $y = ln(-x)$ (above) and $y' = \dfrac{1}{x}$ (below).