18.01 Final Answers

1. (1a) By the product rule,

$$(x^3e^x)' = 3x^2e^x + x^3e^x = e^x(3x^2 + x^3).$$

(1b) If $f(x) = \sin(2x)$, then

$$f^{(7)}(x) = -128\cos(2x)$$

since:

$$f^{(1)}(x) = 2\cos(2x)$$

$$f^{(2)}(x) = -4\sin(2x)$$

$$f^{(3)}(x) = -8\cos(2x)$$

$$f^{(4)}(x) = 16\sin(2x)$$

$$f^{(5)}(x) = 32\cos(2x)$$

$$f^{(6)}(x) = -64\sin(2x)$$

$$f^{(7)}(x) = -128\cos(2x)$$

2. (2a) The line tangent to $y = 3x^2 - 5x + 2$ at $x = 2$ has a slope equal to that of the curve at $x = 2$ and passes through the point (2, 4).

The slope of the line at $x = 2$ is $y'(x = 2) = 6x - 5 = 6(2) - 5 = 7 = m$. The y-intercept of the line, b, is found by using the slope and the known point: $\frac{4-b}{2-0} = 7 \Rightarrow b = -10$.

The equation of the line is therefore

$$y = mx + b = 7x - 10.$$

(2b) If the curve had a horizontal tangent, then at some point the first derivative of y with respect to x would be equal to zero.

The derivative of the equation $xy^3 + x^3y = 4$ is

$$y^3 + x(3y^2)y' + 3x^2y + y'x^3 = 0 \Rightarrow y'(x3y^2 + x^3) = -y^3 - 3x^2y.$$

If y' were equal to 0, then $\dfrac{-y^3 - 3x^2y}{x3y^2 + x^3} = 0 \Rightarrow -y^3 - 3x^2y = 0$. This equation is valid when both x and y are zero or when $y^3 = -3x^2y$ for nonzero x and y.

The first case is not valid, because we are given that $xy^3 + x^3y = 4$, which would not be possible if x and y were both zero.

The second case is also impossible, because $y^3 = -3x^2y \Rightarrow y^2 = -3x^2$ (we can divide by y because in this case it must be nonzero) and it is not possible for the ratio of two squares (necessarily positive numbers) to be equal to a negative number.

Therefore y' can never be zero and so the curve defined by $xy^3 + x^3y = 4$ has no horizontal tangents.

3. (3a)

$$\begin{aligned}
\frac{d}{dx}\left(\frac{x}{x+1}\right) &= \lim_{t \to x} \frac{f(t) - f(x)}{t - x} \\
&= \lim_{t \to x} \frac{\frac{t}{t+1} - \frac{x}{x+1}}{t - x} \\
&= \lim_{t \to x} \frac{t(x+1) - x(t+1)}{(t-x)(t+1)(x+1)} \\
&= \lim_{t \to x} \frac{tx + t - tx - x}{(t-x)(t+1)(x+1)} \\
&= \lim_{t \to x} \frac{t - x}{(t-x)(t+1)(x+1)} \\
&= \lim_{t \to x} \frac{1}{(t+1)(x+1)} \\
&= \frac{1}{(x+1)^2}
\end{aligned}$$

(3b)

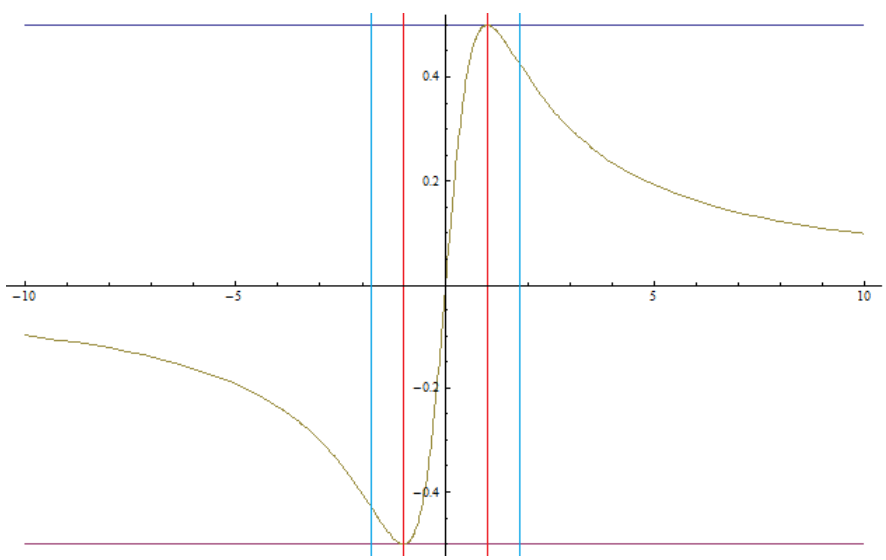
$$\lim_{x \to \sqrt{3}} \frac{\tan^{-1}(x) - \pi/3}{x - \sqrt{3}}$$

When $x \to \sqrt{3}$, the numerator becomes $\pi/3 - \pi/3 = 0$ and as the denominator also goes to zero, we can use l'Hospital's rule to compute the limit:

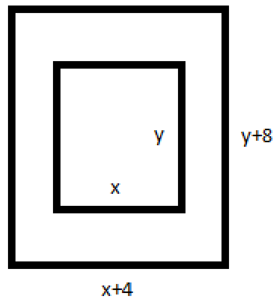
$$\lim_{x\to\sqrt{3}} \frac{(\tan^{-1}(x) - \pi/3)'}{(x - \sqrt{3})'} = \lim_{x\to\sqrt{3}} \frac{1/(1+x^2)}{1}$$
$$= \lim_{x\to\sqrt{3}} \frac{1}{1+x^2}$$
$$= \frac{1}{1+(\sqrt{3})^2}$$
$$= \frac{1}{4}$$

4. As shown in the graph below, $y = \dfrac{x}{x^2+1}$ has the following properties:

- Local maximum ($y' = 0, y'' < 0$) at x=1
- Local minimum ($y' = 0, y'' > 0$) at x=-1
- The function is increasing ($y' > 0$) when $|x| < 1$
- The function is decreasing ($y' < 0$) when $|x| > 1$
- The inflection points ($y'' = 0$) are $x = 0, \pm\sqrt{3}$
- The graph is symmetric about the origin
- The horizontal asymptote $\left(\lim_{x\to\infty} \dfrac{x}{x^2+1}\right)$ is the line $y = 0$
- There is no vertical asymptote



5. The values x and y are defined as in the figure below:



The area of printed type = 50 in^2, so $xy = 50$ and the total area of the poster is $(x+4)(y+8)$. To minimize the amount of paper used, we need to minimize the total area of the poster.

$$(x+4)(y+8) = xy + 4y + 8x + 32 = 82 + 4y + 8x$$

since we know that $xy = 50$.

We can also substitute $y = 50/x$, so that we have an area equal to:

$$82 + \frac{4(50)}{x} + 8x.$$

To find the minimum of this equation we set the first derivative with respect to x equal to zero:

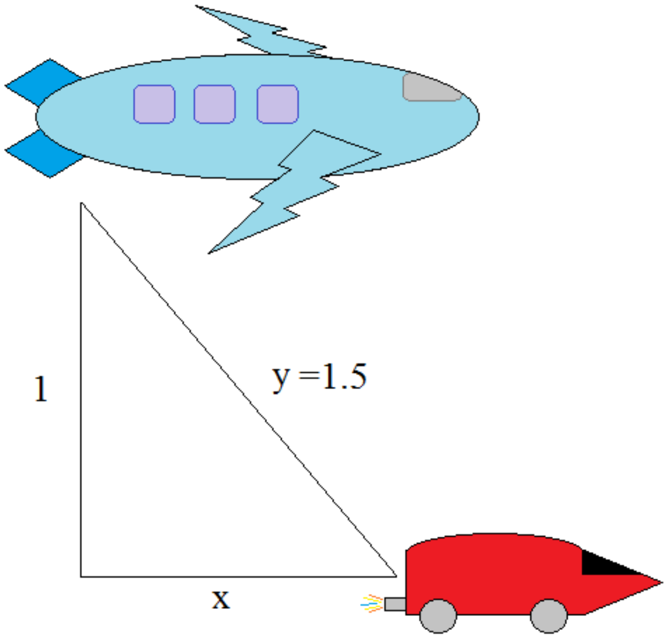
$$-\frac{200}{x^2} + 8 = 0 \Rightarrow x^2 = 25 \Rightarrow x = 5,$$

taking only the positive root because x represents a physical quantity.

We can check that $x = 5$ corresponds to a minimum of the area by taking the second derivative of $-\frac{200}{x^2} + 8$, which is $\frac{400}{x^3}$. Since this is positive at $x = 5$, the point does indeed correspond to a minimum.

If $x = 5$ then $xy = 50 \Rightarrow y = 10$. Thus the dimensions of the poster which minimize the amount of paper used are $a = x + 4 = 9$ in and $b = y + 8 = 18$ in.

6. Let y be the total distance from the plane to the car, and let x be the horizontal distance between the plane and the car. The question asks for dc/dt, the car's speed.



From the Pythagorean theorem, $y = \sqrt{x^2+1}$, because the plane is a distance one mile above the road. By definition, we also know that $dc/dt = dx/dt - 120$, as the plane has speed 120 mph with respect to the ground. In addition, since $y = 3/2$ at $t = 0$, we know that $x = \sqrt{y^2-1} = \frac{\sqrt{5}}{2}$ at $t = 0$.

We can then determine that:

$$\frac{dy}{dt} = \frac{1}{2}(x^2+1)^{-1/2}(2x)\left(\frac{dx}{dt}\right) = -136$$

and we can substitute $x = \sqrt{5}/2$ to obtain:

$$\frac{dx}{dt} = -136\left(\frac{3}{\sqrt{5}}\right) \approx -\frac{408}{2.2}$$

From this we can calculate:

$$dc/dt = \frac{408}{2.2} - 120 \approx 65.5 \text{ mph}$$

7. (7a)

$$\lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{1+\frac{2i}{n}}\left(\frac{2}{n}\right) = \int_0^2 \sqrt{1+x}\,dx$$

$$= \frac{2}{3}(1+x)^{3/2}\Big|_0^2$$

$$= \frac{2}{3}(3)^{3/2} - \frac{2}{3}$$

$$= 2\sqrt{3} - \frac{2}{3}$$

(7b)

$$\lim_{h\to 0}\frac{1}{h}\int_2^{2+h}\sin(x^2)dx = \lim_{h\to 0}\frac{\int_2^{2+h}\sin(x^2)dx}{h}$$

By l'Hospital's rule, this is equal to

$$\lim_{h\to 0}\sin((2+h)^2) = \sin(4)$$

8. (8a)

$$\int_0^{\pi/4}\tan x\sec^2 x dx = \int_0^{\pi/4}\left(\frac{\sin x}{\cos x}\right)\frac{1}{\cos^2 x}dx = \int_0^{\pi/4}\frac{\sin x}{\cos^3 x}dx$$

Let $u = \cos x$. Then $\frac{du}{dx} = -\sin(x)$. Substituting into the integral,

$$\int_0^{\pi/4}\frac{\sin x}{\cos^3 x}dx = -\int_{x=0}^{x=\pi/4}\frac{du}{u^3} = \frac{1}{2}\cos(x)^{-2}\Big|_0^{\pi/4} = \frac{1}{2}\left(\cos(\pi/4)^{-2} - 1\right) = \frac{1}{2}.$$

(8b) Using integration by parts,

$$\int_1^2 x\ln x dx = \frac{1}{2}x^2\ln x\Big|_1^2 - \int_1^2\frac{1}{2}x\ dx$$

$$= \frac{1}{2}(4)\ln(2) - \frac{1}{2}\ln(1) - \frac{1}{4}x^2\Big|_1^2$$

$$= 2\ln(2) - \frac{1}{2}\ln(1) - \frac{3}{4}$$

9. Using the inverse trigonometric substitutions $x = 3\sin\theta, dx = 3\cos\theta d\theta$, the integral becomes

$$\int \frac{9\sin^2\theta(3\cos\theta d\theta)}{\sqrt{9-9\sin^2\theta}} = 9\int \sin^2\theta d\theta.$$

We can then use the double angle formula $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$ to obtain

$$\frac{9}{2}\int (1-\cos 2\theta)\, d\theta.$$

Evaluating the integral, we have

$$\frac{9}{2}\theta - \frac{9}{4}\sin 2\theta + C,$$

where C is a constant of integration. Substituting x back in,

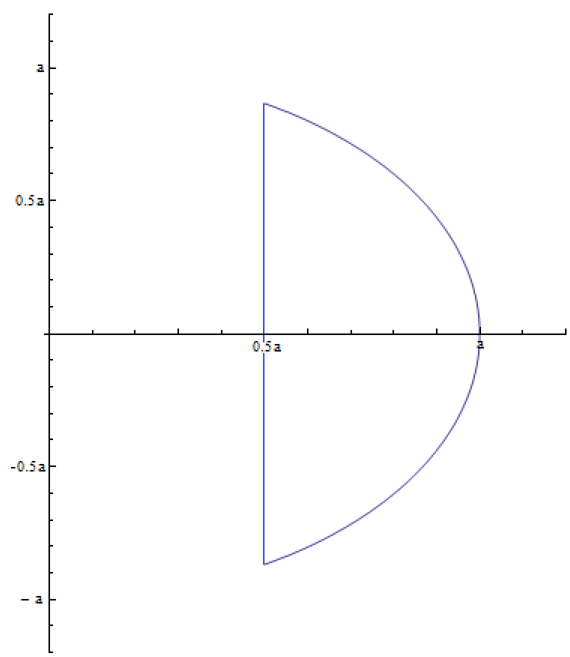
$$\int \frac{x^2 dx}{\sqrt{9-x^2}} = \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{1}{2}x\sqrt{9-x^2} + C$$

*for reference, this is worked out in lec 25, fall 2005, p.4

10. In general, the volume of an area revolved around the y-axis can be found by

$$V = 2\pi \int_a^b x f(x) dx$$

In this case, we are revolving the region as shown in the figure below:



Applying the formula to the region between $\sqrt{a^2 - x^2}, -\sqrt{a^2 - x^2}, x = a$, and $x = a/2$, we obtain:

$$V = 2\pi \int_{a/2}^{a} x2\sqrt{a^2 - x^2}dx$$

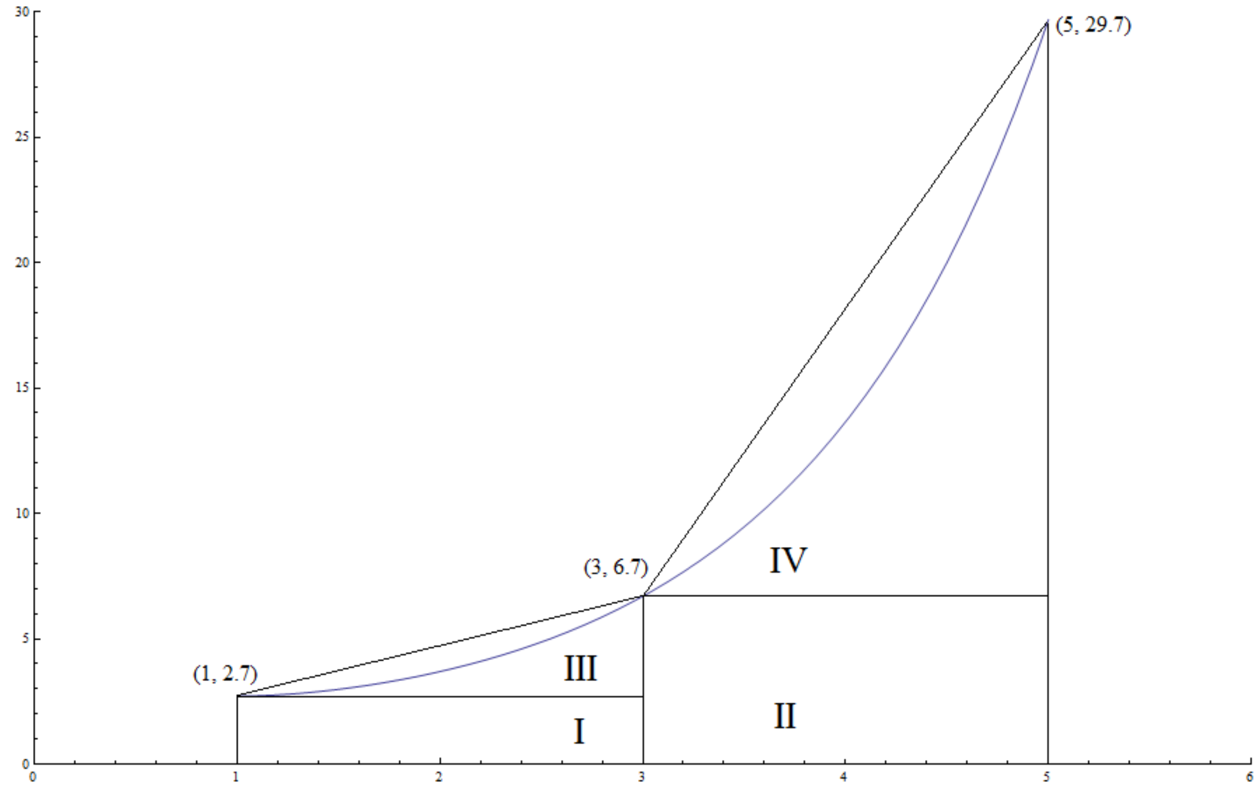
Substituting $u = x^2$ and $du/dx = 2x$:

$$V = 2\pi \int_{x=a/2}^{x=a} \sqrt{a^2 - u}du = 2\pi \left(-\frac{2}{3}(a^2 - u)^{3/2} \right) \Bigg|_{x=a/2}^{x=a}$$

Replacing u with x^2 :

$$\begin{aligned} V &= -\frac{4\pi}{3} \left((a^2 - x^2)^{3/2} \right) \Bigg|_{a/2}^{a} \\ &= -\frac{4\pi}{3} \left(0 - (a^2 - (a/2)^2)^{3/2} \right) \\ &= \frac{4\pi}{3} \left(\frac{3a^2}{4} \right)^{3/2} \\ &= \frac{\sqrt{3}\pi a^3}{2} \end{aligned}$$

11. Let $y(x) = \frac{e^x}{x}$. Using the two-trapezoid method, the picture should be approximately as follows:



The areas of the regions are then:
Region I: $(3-1)y(1) = 2y(1) = 2(2.7) = 5.4$
Region II: $(5-3)y(3) = 2y(3) = 2(6.7) = 13.4$
Region III: $(.5)(3-1)(y(3)-y(1)) = y(3)-y(1) = 6.7-2.7 = 4$
Region IV:$(.5)(5-3)(y(5)-y(3)) = y(5)-y(3) = 29.7-6.7 = 23$
And the total area is then 45.8 units^2.

12. (12a) It is given that the rate of radioactive decay of a mass of Radium-226, dm/dt, is proportional to the amount m of Radium present at time t. We can then write

$$\frac{dm}{dt} = Am,$$

where A is a constant. Re-writing and integrating the equation,

$$\int \frac{dm}{m} = \int Adt$$
$$\ln(m) = At + C'$$
$$m = e^{At+C'} = e^{At}e^{C'}$$
$$m = Ce^{At}$$

where C is a constant. We can find A and C by using the information given in the problem. First, we know that there are 100 mg of Radium present at $t = 0$, so that

$$m(t = 0) = C = 100 \text{ mg}.$$

We also know that it takes 1600 years for m to decrease by half. Therefore:

$$\begin{aligned} (50/100) = .5 &= e^{1600A} \\ \ln(.5) &= 1600A \\ A &= \ln(.5)/1600. \end{aligned}$$

Finally,

$$\begin{aligned} m &= Ce^{At} \\ &= 100e^{(\ln(.5)/1600)t} \\ &= 100(e^{\ln(.5)})^{t/1600} \\ &= 100(.5)^{t/1600}, \end{aligned}$$

where t is in years and $m(t)$ is in mg.

(12b) When $t = 1000$ years, and using the approximation given in the question,

$$\begin{aligned} m &= 100(.5)^{1000/1600} \\ &= 100(2)^{-10/16} \\ &\approx 100(.65) \\ &= 65\text{mg}. \end{aligned}$$

13. The formula for arc length S of a curve defined by parametric equations $x(t)$ and $y(t)$ is:

$$S = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}dt.$$

In this problem, $x(t)$ is given as

$$\int_0^t \cos(\pi u^2/2)du$$

and

$$y(t) = \int_0^t \sin(\pi u^2/2) du.$$

Their derivatives are

$$x'(t) = \cos\left(\frac{\pi t^2}{2}\right)$$
$$y'(t) = \sin\left(\frac{\pi t^2}{2}\right)$$

Substituting $x'(t)$, $y'(t)$, and the appropriate limits into the formula for arc length results in:

$$\begin{aligned} S &= \int_0^{t_0} \sqrt{\cos^2(\pi t^2/2) + \sin^2(\pi t^2/2)} dt \\ &= \int_0^{t_0} dt \\ &= t\Big|_0^{t_0} \\ &= t_0 \end{aligned}$$

14. (14a) The Taylor series of a function $f(x)$ centered at $x = a$ is

$$f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f^{(2)}(a)(x-a)^2}{2!} + \frac{f^{(3)}(a)(x-a)^3}{3!} + \frac{f^{(4)}(a)(x-a)^4}{4!} + \dots$$

The Taylor series of $\ln(1+x)$ centered at $x = a$ is then

$$\ln(1+a) + \frac{(1+a)^{-1}(x-a)}{1!} + \frac{-(1+a)^{-2}(x-a)^2}{2!} + \frac{2(1+a)^{-3}(x-a)^3}{3!} + \frac{-(2)(3)(1+a)^{-4}(x-a)^4}{4!} + \dots$$

And the Taylor series of $\ln(1+x)$ centered at $a = 0$ is therefore

$$\begin{aligned} \ln(1) + \frac{x}{1!} + \frac{-x^2}{2!} + \frac{2x^3}{3!} + \frac{-(2)(3)x^4}{4!} + \dots &= 0 + \frac{x}{1} + \frac{-x^2}{2} + \frac{x^3}{3} + \frac{-x^4}{4} + \dots \\ &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \end{aligned}$$

(14b) Using the ratio test,

$$|x| < \left|\frac{c_n}{c_{n+1}}\right| = \left|\frac{(-1)^{n+1}n}{(-1)^{n+2}n+1}\right| = \left|\frac{n}{n+1}\right|.$$

Because n is the index of summation (an increasing integer), $n+1$ is always greater than n and therefore

$$|x| < \left|\frac{n}{n+1}\right| < 1$$

Thus $|x| < 1$ and the radius of convergence is $-1 < x < 1$.

(14c) $\ln(3/2) = \ln(1 + .5)$ can be approximated by the first two non-zero terms of the Taylor series found in (a):

$$\begin{aligned} \ln(1+x) &\approx \frac{x}{1} + \frac{-x^2}{2} \\ &= .5 - \frac{.25}{2} \\ &= \frac{3}{8} \end{aligned}$$

(14d) The upper bound of the error in (c)'s approximation is found using Taylor's inequality for an approximation of n terms:

$$|R_n(x)| \leq M_n \frac{|x^{n+1}|}{(n+1)!},$$

where $x = 1/2$ and $n = 2$. In addition,

$$M_n \geq |f^{(n+1)}(x)| \Rightarrow M_2 \geq \frac{2}{(1+x)^3}$$

for all $|x| \leq 1/2$; the maximum of M_2 in this range is for $x = -1/2$, which gives $M_2 = 16$. Putting these numbers into the above formula,

$$|R_n(.5)| \leq 16\frac{(.5)^3}{3!} = \frac{1}{3}$$

15. We can prove the inequality by showing that the derivatives of the terms satisfy the inequality for $x > 0$ and then by working backwards from there:

$$d\left(\frac{x}{1+x^2}\right) = \frac{1}{1+x^2} - \frac{2x^2}{(1+x^2)^2}, \quad d(\tan^{-1}(x)) = \frac{1}{1+x^2}, \quad d(x) = 1$$

$$\Rightarrow \frac{1}{1+x^2} - \frac{2x^2}{(1+x^2)^2} < \frac{1}{1+x^2} < 1 \text{ for all } x > 0$$

$$\int_0^t \left(\frac{1}{1+x^2} - \frac{2x^2}{(1+x^2)^2}\right) dx < \int_0^t \frac{1}{1+x^2} dx < \int_0^t 1 dx \text{ for all } x > 0$$

$$\frac{t}{1+t^2} < \tan^{-1}(t) < t \text{ for all } t > 0$$

$$\frac{x}{1+x^2} < \tan^{-1}(x) < x \text{ for all } x > 0$$