18.01 EXERCISES

Unit 1. Differentiation

1A. Graphing

1A-1 By completing the square, use translation and change of scale to sketch
a) $y = x^2 - 2x - 1$ b) $y = 3x^2 + 6x + 2$

1A-2 Sketch, using translation and change of scale
a) $y = 1 + |x + 2|$ b) $y = \dfrac{2}{(x-1)^2}$

1A-3 Identify each of the following as even, odd , or neither
a) $\dfrac{x^3 + 3x}{1 - x^4}$ b) $\sin^2 x$
c) $\dfrac{\tan x}{1 + x^2}$ d) $(1 + x)^4$
e) $J_0(x^2)$, where $J_0(x)$ is a function you never heard of

1A-4 a) Show that every polynomial is the sum of an even and an odd function.

b) Generalize part (a) to an arbitrary function $f(x)$ by writing

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) + f(-x)}{2}$$

Verify this equation, and then show that the two functions on the right are respectively even and odd.

c) How would you write $\dfrac{1}{x + a}$ as the sum of an even and an odd function?

1A-5. Find the inverse to each of the following, and sketch both $f(x)$ and the inverse function $g(x)$. Restrict the domain if necessary. (Write $y = f(x)$ and solve for y; then interchange x and y.)
a) $\dfrac{x - 1}{2x + 3}$ b) $x^2 + 2x$

1A-6 Express in the form $A \sin(x + c)$
a) $\sin x + \sqrt{3} \cos x$ b) $\sin x - \cos x$

1A-7 Find the period , amplitude , and phase angle, and use these to sketch
a) $3 \sin(2x - \pi)$ b) $-4 \cos(x + \pi/2)$

1A-8 Suppose $f(x)$ is odd and periodic. Show that the graph of $f(x)$ crosses the x-axis infinitely often.

1A-9 a) Graph the function f that consist of straight line segments joining the points $(-1,-1)$, $(1,2)$, $(3,-1)$, and $(5,2)$. Such a function is called piecewise linear.

b) Extend the graph of f periodically. What is its period?

c) Graph the function $g(x) = 3f((x/2) - 1) - 3$.

1B. Velocity and rates of change

1B-1 A test tube is knocked off a tower at the top of the Green building. (For the purposes of this experiment the tower is 400 feet above the ground, and all the air in the vicinity of the Green building was evacuated, so as to eliminate wind resistance.) The test tube drops $16t^2$ feet in t seconds. Calculate

a) the average speed in the first two seconds of the fall

b) the average speed in the last two seconds of the fall

c) the instantaneous speed at landing

1B-2 A tennis ball bounces so that its initial speed straight upwards is b feet per second. Its height s in feet at time t seconds is given by $s = bt - 16t^2$

a) Find the velocity $v = ds/dt$ at time t.

b) Find the time at which the height of the ball is at its maximum height.

c) Find the maximum height.

d) Make a graph of v and directly below it a graph of s as a function of time. Be sure to mark the maximum of s and the beginning and end of the bounce.

e) Suppose that when the ball bounces a second time it rises to half the height of the first bounce. Make a graph of s and of v of both bounces, labelling the important points. (You will have to decide how long the second bounce lasts and the initial velocity at the start of the bounce.)

f) If the ball continues to bounce, how long does it take before it stops?

1C. Slope and derivative

1C-1 a) Use the difference quotient definition of derivative to calculate the rate of change of the area of a disk with respect to its radius. (Your answer should be the circumference of the disk.)

b) Use the difference quotient definition of derivative to calculate the rate of change of the volume of a ball with respect to the radius. (Your answer should be the surface area of the ball.)

1C-2 Let $f(x) = (x-a)g(x)$. Use the definition of the derivative to calculate that $f'(a) = g(a)$, assuming that g is continuous.

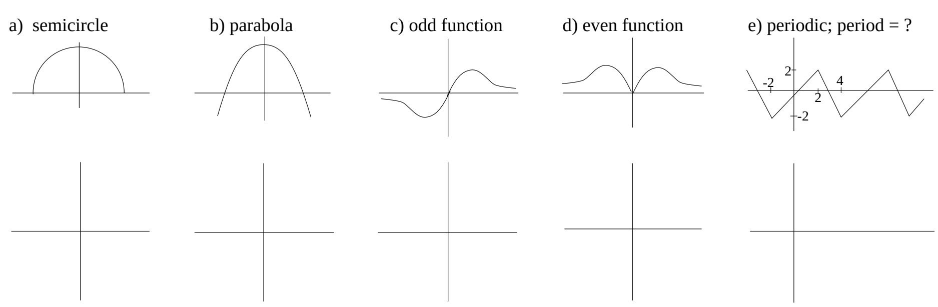
1C-3 Calculate the derivative of each of these functions directly from the definition.

a) $f(x) = 1/(2x+1)$ b) $f(x) = 2x^2 + 5x + 4$
c) $f(x) = 1/(x^2+1)$ d) $f(x) = 1/\sqrt{x}$
e) For part (a) and (b) find points where the slope is $+1$, -1, 0.

1C-4 Write an equation for the tangent line for the following functions
a) $f(x) = 1/(2x+1)$ at $x = 1$ b) $f(x) = 2x^2 + 5x + 4$ at $x = a$
c) $f(x) = 1/(x^2+1)$ at $x = 0$ d) $f(x) = 1/\sqrt{x}$ at $x = a$

1C-5 Find all tangent lines through the origin to the graph of $y = 1 + (x-1)^2$.

1C-6 Graph the derivative of the following functions directly below the graph of the function. It is very helpful to know that the derivative of an odd function is even and the derivative of an even function is odd (see **1F-6**).



1D. Limits and continuity

1D-1 Calculate the following limits if they exist. If they do not exist, then indicate whether they are $+\infty$, $-\infty$ or undefined.

a) $\lim_{x\to 0} \frac{4}{x-1}$ b) $\lim_{x\to 2} \frac{4x}{x+1}$ c) $\lim_{x\to -2} \frac{4x^2}{x+2}$

d) $\lim_{x\to 2^+} \frac{4x^2}{2-x}$ e) $\lim_{x\to 2^-} \frac{4x^2}{2-x}$ f) $\lim_{x\to \infty} \frac{4x^2}{x-2}$

g) $\lim_{x\to \infty} \frac{4x^2}{x-2} - 4x$ i) $\lim_{x\to \infty} \frac{x^2+2x+3}{3x^2-2x+4}$ j) $\lim_{x\to 2} \frac{x-2}{x^2-4}$

1D-2 For which of the following should one use the one-sided limit? Evaluate it.

a) $\lim_{x\to 0} \sqrt{x}$ b) $\lim_{x\to 1} \frac{1}{x-1}$ c) $\lim_{x\to 1} \frac{1}{(x-1)^4}$ d) $\lim_{x\to 0} |\sin x|$ e) $\lim_{x\to 0} \frac{|x|}{x}$

1D-3 Identify and give the type of the points of discontinuity of each of the following:

a) $\frac{x-2}{x^2-4}$ b) $\frac{1}{\sin x}$

c) $\frac{x^4}{x^3}$ d) $f(x) = \begin{cases} x+a, \ x>0 \\ a-x, \ x<0 \end{cases}$

e) $f'(x)$, for the $f(x)$ in d) f) $(f(x))^2$, where $f(x) = \frac{d}{dx}|x|$

1D-4 Graph the following functions.

a) $\frac{4x^2}{x-2}$ (See **1D-1**efg.) b) $\frac{1}{x^2+2x+2}$

1D-5 Define $f(x) = \begin{cases} ax+b, \ x \geq 1; \\ x^2, \ x<1. \end{cases}$

a) Find all values of a, b such that $f(x)$ is continuous.

b) Find all values of a, b such that $f'(x)$ is continuous. (Be careful!)

1D-6 For each of the following functions, find all values of the constants a and b for which the function is differentiable.

a) $f(x) = \begin{cases} x^2+4x+1, \ x \geq 0; \\ ax+b, \ x<0. \end{cases}$ b) $f(x) = \begin{cases} x^2+4x+1, \ x \geq 1; \\ ax+b, \ x<1. \end{cases}$

1D-7 Find the values of the constants a, b and c for which the following function is differentiable. (Give a and b in terms of c.)

$$f(x) = \begin{cases} cx^2 + 4x + 1, \ x \geq 1; \\ ax + b, \ x < 1. \end{cases}$$

1D-8 For each of the following functions, find the values of the constants a and b for which the function is continuous, but *not* differentiable.

a) $f(x) = \begin{cases} ax + b, \ x > 0; \\ \sin 2x, \ x \leq 0. \end{cases}$ b) $f(x) = \begin{cases} ax + b, \ x > 0; \\ \cos 2x, \ x \leq 0. \end{cases}$

1D-9 Find the values of the constants a and b for which the following function is differentiable, but *not* continuous.

$$f(x) = \begin{cases} ax + b, \ x > 0; \\ \cos 2x, \ x \leq 0. \end{cases}$$

1D-10* Show that

$g(h) = \dfrac{f(a+h) - f(a)}{h}$ has a removable discontinuity at $h = 0$ $\iff$ $f'(a)$ exists.

1E. Differentiation formulas: polynomials, products, quotients

1E-1 Find the derivative of the following polynomials.

a) $x^{10} + 3x^5 + 2x^3 + 4$ b) $e^2 + 1$ (e = base of natural logs)
c) $x/2 + \pi^3$ d) $(x^3 + x)(x^5 + x^2)$

1E-2 Find the antiderivative of the following polynomials.

a) $ax + b$ (a and b are constants.) b) $x^6 + 5x^5 + 4x^3$
c) $(x^3 + 1)^2$

1E-3 Find the points (x, y) of the graph $y = x^3 + x^2 - x + 2$ at which the slope of the tangent line is horizontal.

1E-4 For each of the following, find all values of a and b for which $f(x)$ is differentiable.

a) $f(x) = \begin{cases} ax^2 + bx + 4, \ x \leq 0; \\ 5x^5 + 3x^4 + 7x^2 + 8x + 4, \ x > 0. \end{cases}$ b) $f(x) = \begin{cases} ax^2 + bx + 4, \ x \leq 1; \\ 5x^5 + 3x^4 + 7x^2 + 8x + 4, \ x > 1. \end{cases}$

1E-5 Find the derivatives of the following rational functions.

a) $\dfrac{x}{1+x}$ b) $\dfrac{x+a}{x^2+1}$ (a is constant)

c) $\dfrac{x+2}{x^2-1}$ d) $\dfrac{x^4+1}{x}$

1F. Chain rule, implicit differentiation

1F-1 Find the derivative of the following functions:

a) $(x^2+2)^2$ (two methods)

b) $(x^2+2)^{100}$. Which of the two methods from part (a) do you prefer?

1F-2 Find the derivative of $x^{10}(x^2+1)^{10}$.

1F-3 Find dy/dx for $y = x^{1/n}$ by implicit differentiation.

1F-4 Calculate dy/dx for $x^{1/3} + y^{1/3} = 1$ by implicit differentiation. Then solve for y and calculate y' using the chain rule. Confirm that your two answers are the same.

1F-5 Find all points of the curve(s) $\sin x + \sin y = 1/2$ with horizontal tangent lines. (This is a collection of curves with a periodic, repeated pattern because the equation is unchanged under the transformations $y \to y + 2\pi$ and $x \to x + 2\pi$.)

1F-6 Show that the derivative of an even function is odd and that the derivative of an odd function is even.

(Write the equation that says f is even, and differentiate both sides, using the chain rule.)

1F-7 Evaluate the derivatives. Assume all letters represent constants, except for the independent and dependent variables occurring in the derivative.

a) $D = \sqrt{(x-a)^2 + {y_0}^2}$, $\dfrac{dD}{dx} = ?$ b) $m = \dfrac{m_0}{\sqrt{1 - v^2/c^2}}$, $\dfrac{dm}{dv} = ?$

c) $F = \dfrac{mg}{(1+r^2)^{3/2}}$, $\dfrac{dF}{dr} = ?$ d) $Q = \dfrac{at}{(1+bt^2)^3}$, $\dfrac{dQ}{dt} = ?$

1F-8 Evaluate the derivative by implicit differentiation. (Same assumptions about the letters as in the preceding exercise.)

a) $V = \frac{1}{3}\pi r^2 h, \quad \frac{dr}{dh} = ?$ b) $PV^c = nRT, \quad \frac{dP}{dV} = ?$

c) $c^2 = a^2 + b^2 - 2ab\cos\theta, \quad \frac{da}{db} = ?$

1G. Higher derivatives

1G-1 Calculate y'' for the following functions.

a) $3x^2 + 2x + 4\sqrt{x}$ b) $\frac{x}{x+5}$

c) $\frac{-5}{x+5}$ d) $\frac{x^2+5x}{x+5}$

1G-2 Find all functions $f(x)$ whose third derivative $f'''(x)$ is identically zero. ("Identically" is math jargon for "always" or "for every value of x".)

1G-3 Calculate y'' using implicit differentiation and simplify as much as possible.

$$x^2a^2 + y^2b^2 = 1$$

1G-4 Find the formula for the nth derivative $y^{(n)}$ of $y = 1/(x+1)$.

1G-5 Let $y = u(x)v(x)$.

a) Find y', y'', and y'''.

b) The general formula for $y^{(n)}$, the n-th derivative, is called *Leibniz' formula*: it uses the same coefficients as the binomial theorem , and looks like

$$y^{(n)} = u^{(n)}v + \binom{n}{1}u^{(n-1)}v^{(1)} + \binom{n}{2}u^{(n-2)}v^{(2)} + ... + uv^{(n)}$$

Use this to check your answers in part (a), and use it to calculate $y^{(p+q)}$, if $y = x^p(1+x)^q$.

1H. Exponentials and Logarithms: Algebra

1H-1 The *half-life* λ of a radioactive substance decaying according to the law $y = y_0e^{-kt}$ is defined to be the time it takes the amount to decrease to 1/2 of the initial amount y_0.

a) Express the half-life λ in terms of k. (Do this from scratch — don't just plug into formulas given here or elsewhere.)

b) Show using your expression for λ that if at time t_1 the amount is y_1, then at time $t_1 + \lambda$ it will be $y_1/2$, no matter what t_1 is.

1H-2 If a solution containing a heavy concentration of hydrogen ions (i.e., a strong acid) is diluted with an equal volume of water, by approximately how much is its pH changed? (Express $(\text{pH})_{\text{diluted}}$ in terms of $(\text{pH})_{\text{original}}$.)

1H-3 Solve the following for y:

a) $\ln(y+1) + \ln(y-1) = 2x + \ln x$ b) $\log(y+1) = x^2 + \log(y-1)$

c) $2\ln y = \ln(y+1) + x$

1H-4 Solve $\dfrac{\ln a}{\ln b} = c$ for a in terms of b and c; then repeat, replacing ln by log.

1H-5 Solve for x (hint: put $u = e^x$, solve first for u):

a) $\dfrac{e^x + e^{-x}}{e^x - e^{-x}} = y$ b) $y = e^x + e^{-x}$

1H-6 Evaluate from scratch the number $A = \log e \cdot \ln 10$. Then generalize the problem, and repeat the evaluation.

1H-7 The decibel scale of loudness is

$$L = 10\log_{10}(I/I_0)$$

where I, measured in watts per square meter, is the intensity of the sound and $I_0 = 10^{-12}$ watt/m^2 is the softest audible sound at 1000 hertz. Classical music typically ranges from 30 to 100 decibels. The human ear's pain threshold is about 120 decibels.

a) Suppose that a jet engine at 50 meters has a decibel level of 130, and a normal conversation at 1 meter has a decibel level of 60. What is the ratio of the intensities of the two sounds?

b) Suppose that the intensity of sound is proportional to the inverse square of the distance from the sound. Based on this rule, calculate the decibel level of the sound from the jet at a distance of 100 meters, at distance of 1 km.[1]

[1]The inverse square law is justified by the fact that the intensity is measured in energy per unit time per unit area. When the sound has travelled a distance r, the energy of a sound spread over a sphere of radius r centered at the source. The area of that sphere is proportional to r^2, so the average intensity is proportional to $1/r^2$. Fortunately for people who live near airports, sound

1H-8* The mean distance of each of the planets to the Sun and their mean period of revolution is as follows.[2] (Distance is measured in millions of kilometers and time in Earth years.)

9Mercury	Venus	Earth	Mars	Jupiter	Saturn	Uranus	Neptune	Pluto
57.9	108	150	228	778	1,430	2,870	4,500	5,900
0.241	0.615	1.00	1.88	11.9	29.5	84.0	165	248

a) Find the pattern in these data by making a graph of $(\ln x, \ln y)$ where x is the distance to the Sun and y is the period of revolution for the first four points (Mercury through Mars). Observe that these points are nearly on a straight line. Plot a line with ruler and estimate its slope. (You can check your estimated slope by calculating slopes of lines connecting consecutive data points.)

b) Using an approximation to the slope m that you found in part (a) accurate to two significant figures, give a formula for y in the form

$$\ln y = m \ln x + c$$

(Use the Earth to evaluate c.)

c) Solve for y and make a table for the predicted values of the periods of revolutions of all the planets based on their distance to the Sun. (Your answers should be accurate to one percent.)

d) The Earth has radius approximately $6,000$ km and the Moon is at a distance of about $382,000$ km. The period of revolution of the Moon is a lunar month, say 28 days. Assume that the slope m is the same for revolution around the Earth as the on you found for revolution around the Sun in (a). Find the distance above the surface of the Earth of geosynchronous orbit, that is, the altitude of the orbit of a satellite that stays above one place on the equator. (For satellites this close to Earth it is important to know that y is predicting the distance from the satellite to the center of the Earth. This is why you need to know the radius of the Earth.)

e) Find the period of revolution of a satellite that circles at an altitude of $1,000$ km.

1I. Exponential and Logarithms: Calculus

1I-1 Calculate the derivatives
a) xe^x b) $(2x-1)e^{2x}$ c) e^{-x^2}

doesn't travel as well as this. Part of the energy is dissipated into heating the air and another part into vibration of insulating materials on the way to the listener's ear.

[2]from "Fundamentals of Physics, vol. 1," by D. Halliday and R. Resnick

d) $x\ln x - x$ e) $\ln(x^2)$ f) $(\ln x)^2$
g) $(e^{x^2})^2$ h) x^x i) $(e^x + e^{-x})/2$
j) $(e^x - e^{-x})/2$ k) $\ln(1/x)$ l) $1/\ln x$
m) $(1 - e^x)/(1 + e^x)$

1I-2 Graph the function $y = (e^x + e^{-x})/2$.

1I-3 a) Evaluate $\lim_{n\to\infty} n\ln(1 + \frac{1}{n})$.
Hint: Let $h = 1/n$, and use $(d/dx)\ln(1 + x)|_{x=0} = 1$.

b) Deduce that $\lim_{n\to\infty}(1 + \frac{1}{n})^n = e$.

1I-4 Using $\lim_{n\to\infty}(1 + \frac{1}{n})^n = e$, calculate

a) $\lim_{n\to\infty}(1 + \frac{1}{n})^{3n}$ b) $\lim_{n\to\infty}(1 + \frac{2}{n})^{5n}$ c) $\lim_{n\to\infty}(1 + \frac{1}{2n})^{5n}$

1I-5* If you invest P dollars at the annual interest rate r, then after one year the interest is $I = rP$ dollars, and the total amount is $A = P + I = P(1 + r)$. This is *simple interest.*

For *compound interest*, the year is divided into k equal time periods and the interest is calculated and added to the account at the end of each period. So at the end of the first period, $A = P(1 + r(\frac{1}{k}))$; this is the new amount for the second period, at the end of which $A = P(1 + r(\frac{1}{k}))(1 + r(\frac{1}{k}))$, and continuing this way, at the end of the year the amount is

$$A = P\left(1 + \frac{r}{k}\right)^k .$$

The compound interest rate r thus earns the same in a year as the simple interest rate of

$$\left(1 + \frac{r}{k}\right)^k - 1 ;$$

this equivalent simple interest rate is in bank jargon the "annual percentage rate" or APR.[3]

a) Compute the APR of 5% compounded monthly, daily,[4] and continuously. Continuous compounding means the limit as k tends to infinity.

b) As in part (a), compute the APR of 10% compounded monthly, biweekly (k=26), daily, and continuously. (We have thrown in the biweekly rate because loans can be paid off biweekly.)

[3]Banks are required to reveal this so-called APR when they offer loans. The APR also takes into account certain bank fees known as points. Unfortunately, not all fees are included in it, and the true costs are higher if the loan is paid off early.

[4]For daily compounding assume that the year has 365 days, not 365.25. Banks are quite careful about these subtle differences. If you look at official tables of rates from precalculator days you will find that they are off by small amounts because U.S. regulations permitted banks to pretend that a year has 360 days.

1J. Trigonometric functions

1J-1 Calculate the derivatives of the following functions

a) $\sin(5x^2)$ b) $\sin^2(3x)$ c) $\ln(\cos(2x))$
d) $\ln(2\cos x)$ e) $\dfrac{\sin x}{x}$ f) $\cos(x+y)$; $y = f(x)$
g) $\cos(x+y)$; y constant h) $e^{\sin^2 x}$ i) $\ln(x^2 \sin x)$
j) $e^{2x}\sin(10x)$ k) $\tan^2(3x)$ l) $\sec\sqrt{1-x^2}$
m) The following three functions have the same derivative: $\cos(2x)$, $\cos^2 x - \sin^2 x$, and $2\cos^2 x$. Verify this. Are the three functions equal? Explain.
n) $\sec(5x)\tan(5x)$ o) $\sec^2(3x) - \tan^2(3x)$ p) $\sin(\sqrt{x^2+1})$
q) $\cos^2(\sqrt{1-x^2})$ r) $\tan^2(\dfrac{x}{x+1})$

1J-2 Calculate $\displaystyle\lim_{x\to\pi/2} \frac{\cos x}{x-\pi/2}$ by relating it to a value of $(\cos x)'$.

1J-3 a) Let $a > 0$ be a given constant. Find in terms of a the value of $k > 0$ for which $y = \sin(kx)$ and $y = \cos(kx)$ both satisfy the equation

$$y'' + ay = 0.$$

Use this value of k in each of the following parts.

b) Show that $y = c_1 \sin(kx) + c_2 \cos(kx)$ is also a solution to the equation in (a), for any constants c_1 and c_2.

c) Show that the function $y = \sin(kx + \phi)$ (whose graph is a sine wave with phase shift ϕ) also satisfies the equation in (a), for any constant ϕ.

d) Show that the function in (c) is already included among the functions of part (b), by using the trigonometric addition formula for the sine function. In other words, given k and ϕ, find values of c_1 and c_2 for which

$$\sin(kx+\phi) = c_1\sin(kx) + c_2\cos(kx)$$

1J-4 a) Show that a chord of the unit circle with angle θ has length $\sqrt{2-2\cos\theta}$. Deduce from the half-angle formula

$$\sin(\theta/2) = \sqrt{\frac{1-\cos\theta}{2}}$$

that the length of the chord is

$$2\sin(\theta/2)$$

b) Calculate the perimeter of an equilateral n-gon with vertices at a distance 1 from the center. Show that as n tends to infinity, the perimeter tends to 2π, the circumference of the unit circle.