Example: Reciprocals

Let's use the quotient rule in a simple example. The quotient rule tells us that:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2}$$

In this example u will be 1, so we'll be finding the derivative of $\frac{1}{v}$, the reciprocal of v.

$$\frac{d}{dx}\left(\frac{1}{v}\right) = ?$$

We're going to use the formula above. We know $u = 1$ and $v = v$, so we still need to find $\frac{du}{dx}$ and $\frac{dv}{dx}$ before we can apply the formula.

The derivative of a constant (like 1) is zero, so $\frac{du}{dx} = 0$. We don't know what v is, so we'll just write $\frac{dv}{dx} = v'$. Plugging all this in to the quotient rule formula we get:

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{v}\right) &= \frac{0 \cdot v - 1v'}{v^2} \\ &= \frac{-v'}{v^2} \\ &= -v^{-2}v'\end{aligned}$$

Now we have a general formula that lets us differentiate reciprocals! Next, let's use this formula to see what happens when $u = 1$ and $v = x^n$. Here again $\frac{du}{dx} = 0$ and now $v' = \frac{d}{dx}x^n = nx^{n-1}$.

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{x^n}\right) &= -v^{-2}v' \\ &= -(x^n)^{-2}(nx^{n-1}) \\ &= -x^{-2n}(nx^{n-1}) \\ &= -nx^{-n-1}\end{aligned}$$

But $\frac{1}{x^n} = x^{-n}$, which is x to a power. We have a rule for taking the derivative of x to a positive power; how does that compare to our new rule for the derivative of x to a negative power?

$$\frac{d}{dx}x^{-n} = -nx^{-n-1}$$

This agrees with the formula $\frac{d}{dx}x^n = nx^{n-1}$, so the quotient rule confirms that our rule for taking the derivative of x^n works even when n is negative.