

**PROFESSOR:** Welcome to recitation. Today in this video what we're going to do is look at how we can determine the graph of a derivative of a function from the graph of the function itself.

So I've given a function here. We're calling it just  $y$  equals  $f$  of  $x$ -- or this is the curve,  $y$  equals  $f$  of  $x$ . So we're thinking about a function  $f$  of  $x$ . I'm not giving you the equation for the function. I'm just giving you the graph. And what I'd like you to do, what I'd like us to do in this time, is to figure out what the curve  $y$  equals  $f$  prime of  $x$  will look like. So that's our objective.

So what we'll do first is try and figure out the things that we know about  $f$  prime of  $x$ . So what I want to remind you is that when you think about a function's derivative, remember its derivative's output is measuring the slope of the tangent line at each point. So that's what we're interested in finding, is understanding the slope of the tangent line of this curve at each  $x$ -value.

So it's always easiest when you're thinking about a derivative to find the places where the slope of the tangent line is 0. Because those are the only places where you can hope to change the sign on the derivative. So what we'd like to do is first identify, on this curve, where the tangent line has slope equal to 0. And I think there are two places we can find it fairly easily. That would be at whatever this  $x$  value is, that slope there is 0. It's going to be a horizontal tangent line. And then whatever this  $x$  value is. The slope there is also 0. Horizontal tangent line.

But there's a third place where the slope of the tangent line is 0, and that's kind of hidden right in here. And actually, I've drawn in-- maybe you think there are a few more-- but we're going to assume that this function is always continuing down through this region. So there are three places where the tangent line is horizontal. So I can even sort of draw them lightly through here. You have three horizontal tangent lines.

So at those points, we know that the derivative's value is equal to 0, the output is equal to 0. And now what we can determine is, between those regions, where are the values of the derivative positive and negative?

So what I'm going to do is below here, I'm just going to make a line and we're going to sort of keep track of what the signs of the derivative are. So let me just draw. This would be sort of our sign on  $f$  prime. OK. So that's going to tell us what our signs are. So right below, we'll keep

track.

So here, this, I'll just come straight down. Here we know the sign of  $f'$  is equal to 0. OK? We know it's equal to 0 there. We know it's also equal to 0 here, and we know it's also equal to 0 here. OK?

And now the question is, what is the sign of  $f'$  in this region? So to the left of whatever that  $x$  value is. What is the sign of  $f'$  in this region, in this region, and then to the right? So there are really-- we can divide up the  $x$ -values as left of whatever that  $x$ -value is, in between these two values, in between these two values, and to the right of this  $x$ -value. That's really, really what we need to do to determine what the signs of  $f'$  are.

So again, what we want to do to understand  $f'$  is we look at the slope of the tangent line of the curve  $y = f(x)$ . So let's pick a place in this region left of where it's 0, say right here, and let's look at the tangent line. The tangent line has what kind of slope? Well, it has a positive slope. And in fact, if you look along here, you see all of the slopes are positive. So  $f'$  is bigger than 0 here. And now I'm just going to record that. I'm going to keep that in mind as a plus. The sign is positive there.

Now, if I look right of where  $f'$  equals 0, if I look for  $x$ -values to the right, I see that as I move to the right, the tangent line is curving down. So let me do it with the chalk. You see the tangent line looks, has a slope negative slope. If I draw one point in, it looks something like that. So the slope is negative there. So here I can record that. The sign of  $f'$  is a minus sign there.

Now, if I look between these two  $x$ -values, which I'm saying here it's 0 and here it's 0 for the  $x$  values, and I take a take a point, we notice the sign is negative there, also. So in fact, the sign of  $f'$  changed at this zero of  $f'$ , but it stays the same around this zero of  $f'$ . So it's negative and then it goes to negative again. It's negative, then 0, then negative.

And then if I look to the right of this  $x$ -value and I take a point, I see that the slope of the tangent line is positive. And so the sign there is positive.

So we have the derivative is positive, and then 0, and then negative, and then 0, and then negative, and then 0, and then positive. So there's a lot going on. But I, if I want to plot, now,  $y = f'(x)$ , I have some sort of launching point by which to do that.

So what I can do is, I know that the derivative 0-- I'm going to draw the derivative in blue, here--

- the derivative is 0, its output is 0 at these places. So I'm going to put those points on. And then if I were just trying to get a rough idea of what happens, the derivative is positive left of this  $x$  value. So it's certainly coming down. It's coming down. Oops, let me make these a little darker. It's coming down because it's positive. It's coming down to 0-- it has to stay above the  $x$ -axis, but it has to head towards 0. Right?

What does that actually correspond to? Well, look at what the slopes are doing. The slopes of these tangent lines, as I move in the  $x$ -direction, the slope-- let me just keep my hand, watch what my hand is doing-- the slope is always positive, but it's becoming less and less vertical, right? It's headed towards horizontal.

So the slope that was steeper over here is becoming less steep. The steepness is really the magnitude of the derivative. That's really measuring how far it is, the output is, from 0. So as the derivative becomes less steep, the derivative's values have to be headed closer to 0.

Now, what happens when the derivative is equal to 0 here? Well, all of a sudden the slopes are becoming negative. So the outputs of the derivative are negative. It's going down.

But then once it hits here, again, notice what happens. The derivative is 0 again, and notice how I get there. The derivative's negative, and then it starts to-- the slopes of these tangent lines start to get shallower. Right? They were steep and then somewhere they start to get shallower.

So there's someplace sort of in the  $x$ -values between here and here where the derivative is as steep as it gets in this region, and then gets less steep. The steepest point is that point where you have the biggest magnitude in that region for  $f'$ . So that's where it's going to be furthest from 0. So if I'm guessing, it looks like right around here the tangent line is as steep as it ever gets in that region, between these two zeros, and then it gets less steep. So I'd say, right around there we should say, OK, that's as low as it goes and now it's going to come back up. OK?

So hopefully that makes sense. We'll get to see it again, here.

Between these two zeros the same kind of thing happens. But notice-- this is, we have to be careful-- we shouldn't go through 0 here because the derivative's output, the sign is negative. Right? Notice, so the tangent line, it was negative, negative, negative, 0, oh, it's still negative. So the outputs are still negative, and they're going to be negative all the way to this zero.

And what we need to see again is the same kind of thing happens as happened in this region will happen in this region. The point being that, again, we're 0 here. We're 0 here. So somewhere in the middle, we start at 0, the tangent lines start to get steeper, then at some point they stop getting steeper, they start getting shallower. That place looks maybe right around here. That's the sort of steepest tangent line, then it gets less steep.

So that's the place where the derivative's magnitude is going to be the biggest in this region. And actually, I've sort of drawn it, they look like they're about the same steepness at those two places, so I should probably put the outputs about the same down here. Their magnitudes are about the same. So this has to bounce off, come up here. I made that a little sharper than I meant to. OK?

So that's the place. That's the output here-- or the tangent line, sorry. The tangent line at this  $x$  value is the steepest that we get in this region, so the output at that  $x$ -value is the lowest we get. And then, when we're to the right of this zero for the derivative, we start seeing the tangent lines positive-- we pointed that out already-- and it gets more positive. So it starts at 0, it starts to get positive, and then it gets more positive. It's going to do something like that, roughly.

So let me fill in the dotted lines so we can see it clearly. Well, this is not exact, but this is a fairly good drawing, I think we can say, of  $f'$  of  $x$ .  $y$  equals  $f'$  of  $x$ .

And now I'm going to ask you a question. I'm going to write it on the board, and then I'm going to give you a moment to think about it. So let me write the question.

It's, find a function  $y$  equals-- or sorry-- find a function  $g$  of  $x$  so that  $y$  equals  $g'$  of  $x$  looks like  $y$  equals  $f'$  of  $x$ . OK, let me be clear about that, and then I'll give you a moment to think about it. So I want you to find a function  $g$  of  $x$  so that its derivative's graph,  $y$  equals  $g'$  of  $x$ , looks exactly like the graph we've drawn in blue here,  $y$  equals  $f'$  of  $x$ . Now, I don't want you to find something in terms of  $x$  squareds and  $x$  cubes. I don't want you to find an actual  $g$  of  $x$  equals something in terms of  $x$ . I want you to just try and find a relationship that it must have with  $f$ . So I'm going to give me a moment to think about it and work out your answer, and I'll be back to tell you.

OK. Welcome back.

So what we're looking for is a function  $g$  of  $x$  so that its derivative, when I graph it,  $y$  equals  $g'$  of  $x$ , I get exactly the same curve as the blue one. The blue one. And the point is that if you thought about it for a little bit, what you really need is a function that looks exactly like this function,  $y$  equals  $f$  of  $x$ , at all the  $x$ -values in terms of its slopes, but those slopes can happen shifted up or down anywhere. So the point is that if I take the function  $y$  equals  $f$  of  $x$  and I add a constant to it, which shifts the whole graph up or down, the tangent lines are unaffected by that shift. And so I get exactly the same picture when I take the derivative of that graph. When I look at that the tangent line slopes of that graph.

So you could draw another picture and check it for yourself if you didn't feel convinced, shift this, shift this curve up, and then look at what the tangent lines do on that curve. But then you'll see its derivative's outputs are exactly the same.

So we'll stop there.