1. Compute the following derivatives. (Simplify your answers when possible.)

   (a) $f'(x)$ where $f(x) = \dfrac{x}{1-x^2}$

$$f'(x) = \frac{1(1-x^2) - (x)(-2x)}{(1-x^2)^2} = \frac{1 - x^2 + 2x^2}{(1-x^2)^2} = \frac{1+x^2}{(1-x^2)^2}$$

   (b) $f'(x)$ where $f(x) = \ln(\cos x) - \dfrac{1}{2}\sin^2(x)$

$$\begin{aligned} f'(x) &= \frac{1}{\cos x}(-\sin x) - \frac{1}{2} \cdot 2 \sin x \cos x \\ &= \boxed{-\tan x - \sin x \cos x} \\ &= \boxed{-\sin x \left(\frac{1 + \cos^2 x}{\cos x} \right)} \end{aligned}$$

   (c) $f^{(5)}(x)$, the fifth derivative of f, where $f(x) = xe^x$

$$\begin{aligned} f'(x) = e^x + xe^x &= 1 \cdot e^x + xe^x \\ f''(x) = e^x + e^x + xe^x &= 2 \cdot e^x + xe^x \\ f^{(3)}(x) = 2e^x + e^x + xe^x &= 3 \cdot e^x + xe^x \\ f^{(4)}(x) &= 4 \cdot e^x + xe^x \\ f^{(5)}(x) &= 5 \cdot e^x + xe^x \end{aligned}$$

   The inductive step in the proof of this for the general case looks like:

$$\begin{aligned} f^{(k)}(x) &= ke^x + xe^x \\ \Rightarrow f^{(k+1)}(x) &= ke^x + e^x + xe^x \\ &= (k+1)e^x + xe^x. \end{aligned}$$

2. Find the equation of the tangent line to the "astroid" curve defined implicitly by the equation

$$x^{2/3} + y^{2/3} = 4$$

   at the point $(-\sqrt{27}, 1)$.

   Check that the point is on the curve:

$$-\sqrt{27} = -3^{3/2}.$$

$$(-3^{3/2})^{2/3} + (1)^{2/3} = 3 + 1 = 4.$$

Use implicit differentiation to get $\left.\dfrac{dy}{dx}\right|_{(-\sqrt{27},1)}$

$$\begin{aligned}
x^{2/3} + y^{2/3} &= 4 \\
\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} &= 0 \\
x^{-1/3} + y^{-1/3}\frac{dy}{dx} &= 0 \\
\frac{dy}{dx} &= -\frac{x^{-1/3}}{y^{-1/3}} \\
&= -\frac{(-3^{3/2})^{-1/3}}{1} \\
\frac{dy}{dx} &= 3^{-1/2} = \frac{1}{\sqrt{3}}.
\end{aligned}$$

The slope of the tangent line is $\frac{1}{\sqrt{3}}$. The point-slope formula tells us that the equation of the tangent line is:

$$\begin{aligned}
y - 1 &= \frac{1}{\sqrt{3}}(x + \sqrt{27}) \\
y &= \frac{1}{\sqrt{3}}x + \sqrt{9} + 1 \\
y &= \frac{1}{\sqrt{3}}x + 4.
\end{aligned}$$

3. A particle is moving along a vertical axis so that its position y (in meters) at time t (in seconds) is given by the equation

$$y(t) = t^3 - 3t + 3, \quad t \geq 0.$$

Determine the total distance traveled by the particle in the first three seconds.

Cubic functions tend to increase, then decrease, then increase; we may need to break the journey into three parts to get the total (not net) distance traveled. Thus, we start by finding the max. and min. of $y(t)$.

To find the min/max we set $y'(t) = 0$ and solve for t:

$$\begin{aligned}
y'(t) &= 3t^2 - 3 = 3(t^2 - 1) = 0 \\
\Rightarrow t &= \pm 1.
\end{aligned}$$

Note that $y'(t) > 0$ for $t < -1$ and $t > 1$ and $y'(t) < 0$ for $-1 < t < 1$, so the particle is initially descending and then at $t = 1$ it starts to ascend.

$$\begin{aligned} y(0) &= 3 \\ y(1) &= 1 - 3 + 3 = 1 \\ y(3) &= 27 - 9 + 3 = 21 \end{aligned}$$

The total distance traveled is $\underbrace{(3-1)}_{\text{down}} + \underbrace{(21-1)}_{\text{up}} = 22.$

4. State the product rule for the derivative of a pair of differentiable functions f and g using your favorite notation. Then use the DEFINITION of the derivative to prove the product rule. Briefly justify your reasoning at each step.

   If f and g are both differentiable functions of x, then:

$$(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x).$$

$$\begin{aligned} (f \cdot g)'(x) &= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\ &= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \\ &= \lim_{h \to 0} \left(\frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} + \frac{f(x)g(x+h) - f(x)g(x)}{h} \right) \\ &= \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} g(x+h) + f(x) \frac{g(x+h) - g(x)}{h} \right) \end{aligned}$$

   Because f and g are differentiable they must be continuous, so $\lim_{h \to 0} g(x+h) = g(x)$. Therefore:

$$\begin{aligned} (f \cdot g)'(x) &= \lim h \to 0 \left(\frac{f(x+h) - f(x)}{h} g(x+h) + f(x) \frac{g(x+h) - g(x)}{h} \right) \\ &= f'(x)g(x) + f(x)g'(x). \end{aligned}$$

5. Does there exist a set of real numbers a, b and c for which the function

$$f(x) = \begin{cases} \tan^{-1}(x) & x \leq 0 \\ ax^2 + bx + c, & 0 < x < 2 \\ x^3 - \frac{1}{4}x^2 + 5, & x \geq 2 \end{cases}$$

is differentiable (i.e. everywhere differentiable)? Explain why or why not. (Here $\tan^{-1}(x)$ denotes the inverse of the tangent function.)

We start by finding the constraints on a, b and c under which the function is continuous:

$$\tan^{-1}(0) = 0 \Rightarrow c = 0.$$

At $x = 2$:

$$2^3 - \frac{1}{4}2^2 + 5 = 8 - 1 + 5 = 12.$$

Therefore $a \cdot 2^2 + b \cdot 2 + c = 12, \Rightarrow 4a + 2b = 12, \Rightarrow b = 6 - 2a$.

Next we find the conditions that ensure differentiability at $x = 0$ and $x = 2$.

$$f'(x)|_{x=2} = \left[3x^2 - \frac{1}{2}x\right]_{x=2} = 3 \cdot 4 - \frac{1}{2} \cdot 2 = 11.$$

So we want: $[2ax + b]_{x=2} = 4a + b = 11$.

In order for the function to be continuous we must have $b = 6 - 2a$, so if f is both differentiable and continuous $4a + (6 - 2a) = 11 \Rightarrow a = 5/2, b = 1$.

If f is differentiable then $a = 5/2$, $b = 1$ and $c = 0$. We must check that under these conditions, f is differentiable at $x = 0$.

At $x = 0$ the derivative of $\frac{5}{2}x^2 + x$ is $5 \cdot 0 + 1 = 1$.

If we have forgotten that the derivative of $\tan^{-1}(x)$ is $\dfrac{1}{1+x^2}$ then we must apply implicit differentiation to the function $\tan y = x$ to re-derive this fact, as presented in lecture. At $x = 0$ the derivative of $\tan^{-1}(x)$ is $\frac{1}{1+0} = 1$.

We conclude that when $a = 5/2$, $b = 1$ and $c = 0$, the function $f(x)$ defined above is differentiable.

6. Suppose that f satisfies the equation $f(x + y) = f(x) + f(y) + x^2y + xy^2$ for all real numbers x and y. Suppose further that

$$\lim_{x \to 0} \frac{f(x)}{x} = 1.$$

(a) Find $f(0)$.

Since $\displaystyle\lim_{x \to 0} \frac{f(x)}{x} = 1$, $f(x) \to 0$ as $x \to 0$, so $f(0) = 0$.

(b) Find $f'(0)$.

$$f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h)}{h} = 1.$$

(c) Find $f'(x)$.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x)+f(h)+x^2h+xh^2-f(x)}{h} \\
&= \lim_{h\to 0} \left(\frac{f(h)}{h} + x^2 + xh \right) \\
&= 1 + x^2 + 0 \\
&= \boxed{x^2+1}
\end{aligned}$$