The Chain Rule, Revisited

Why it's true

We didn't fully explain why the chain rule is true. We'll look at an example that should explain that. Consider the function

$$y = 10x + b.$$

Here y is changing ten times as fast as x, which is to say that $\frac{dy}{dx} = 10$.

Now, what if x is also a function of some variable t? If

$$x = 5t + a$$

then $\frac{dx}{dt} = 5$.

The chain rule says that if y is going ten times as fast as x, and x is going five times as fast as t, then y is going fifty times as fast as t. Algebraically, I replace x by $5t$ in the equation for y to get:

$$y = 10x + b = 10(5t + a) + b = 50t + 10a + b.$$

The consequence is that $\frac{dy}{dt} = 50 = 10 \cdot 5 = \frac{dy}{dx}\frac{dx}{dt}$. This is, in a nutshell, why the chain rule works and why these rates multiply.

Things it's good for

The chain rule can also make some of the other rules a little easier to remember or possibly to avoid. The messiest rule is perhaps the quotient rule. Notice that $\left(\frac{1}{v}\right)' = (v^{-1})'$. Instead of using the quotient rule here we can use the chain rule with the power -1 and the power law:

$$\left(\frac{1}{v}\right)' = (v^{-1})' = -v^{-2}v'.$$

Similarly,

$$\left(\frac{u}{v}\right)' = (uv^{-1})' = u'v^{-1} + u(-v^{-2})v'.$$

This explains the minus sign in the formula:

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}.$$