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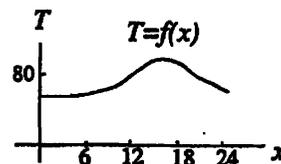
18.01 Single Variable Calculus
Fall 2006

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AV. AVERAGE VALUE

What was the average temperature on July 4 in Boston?

The temperature is a continuous function $f(x)$, whose graph over the 24-hour period might look as shown. How should we define the average value of such a function over the time interval $[0, 24]$ — measuring time x in hours, with $x = 0$ at 12:00AM?



We could observe the temperature in the middle of every hour, that is, at the times $x_1 = .5, x_2 = 1.5, \dots, x_{24} = 23.5$, then average these 24 observations, getting

$$\frac{1}{24} \sum_{i=1}^{24} f(x_i).$$

To get a more accurate answer, we could average measurements made more frequently, say every ten minutes.

For a general interval $[a, b]$ and function $f(x)$, the analogous procedure would be to divide up the interval into n equal parts, each of length

$$(1) \quad \Delta x = \frac{b-a}{n}$$

and average the values of the function $f(x)$ at a succession of points x_i , where x_i lies in the i -th interval. Then we ought to have

$$(2) \quad \text{average of } f(x) \text{ over } [a, b] \approx \frac{1}{n} \sum_{i=1}^n f(x_i).$$

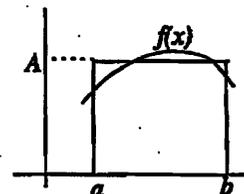
We can relate the sum on the right to a definite integral: using (1), (2) becomes

$$(3) \quad \text{average of } f(x) \text{ over } [a, b] \approx \frac{1}{b-a} \sum_{i=1}^n f(x_i) \Delta x.$$

As $n \rightarrow \infty$, the sum on the right-hand side of (3) approaches the definite integral of $f(x)$ over $[a, b]$, and we therefore define the average value of the function $f(x)$ on $[a, b]$ by

$$(4) \quad A = \text{average of } f(x) \text{ over } [a, b] = \frac{1}{b-a} \int_a^b f(x) dx.$$

Geometrically, the average value A can be thought of as the height of that constant function A which has the same area over $[a, b]$ as $f(x)$ does. This is so since (4) shows that



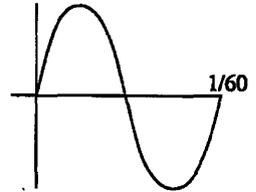
$$A \cdot (b-a) = \int_a^b f(x) dx.$$

Example 1. In alternating current, voltage is represented by a sine wave with a frequency of 60 cycles/second, and a peak of 120 volts. What is the average voltage?

Solution. The voltage function has frequency $\frac{2\pi}{\text{period}} = \frac{2\pi}{1/60} = 120\pi$, and amplitude 120, so it is given by $V(t) = 120 \sin(120\pi t)$. Thus by (4),

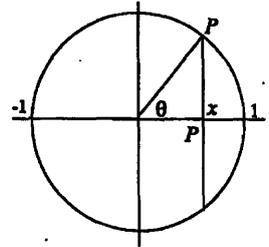
$$\text{average } V(t) = 120 \int_0^{1/120} V(t) dt = -\frac{120}{\pi} \cos(120\pi t) \Big|_0^{1/120} = \frac{2}{\pi} \cdot 120.$$

(We integrate over $[0, 1/120]$ rather than $[0, 1/60]$ since we don't want zero as the average.)



Example 2.

a) A point is chosen at random on the x -axis between -1 and 1 ; call it P . What is the average length of the vertical chord to the unit circle passing through P ?



b) Same question, but now the point P is chosen at random on the circumference.

Solution. a) If P is at x , the chord has length $2\sqrt{1-x^2}$, so we get

$$\text{average of } 2\sqrt{1-x^2} \text{ over } [-1, 1] = \frac{1}{2} \int_{-1}^1 2\sqrt{1-x^2} dx = \text{area of semicircle} = \frac{\pi}{2} \approx 1.6.$$

b) By symmetry, we can suppose P is on the upper semicircle. If P is at the angle θ , the chord has length $2 \sin \theta$, so this time we get

$$\text{average of } 2 \sin \theta \text{ over } [0, \pi] = \frac{1}{\pi} \int_0^\pi 2 \sin \theta d\theta = -\frac{2}{\pi} \cos \theta \Big|_0^\pi = \frac{4}{\pi} \approx 1.3.$$

(Intuitively, can you see why the average in part (b) should be less than the average in part (a) — could you have predicted this would be so?)

Exercises: Section 4D