18.01 Single Variable Calculus
Fall 2006

Lecture 7: Continuation and Exam Review

Hyperbolic Sine and Cosine

Hyperbolic sine (pronounced "sinsh"):

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

Hyperbolic cosine (pronounced "cosh"):

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\frac{d}{dx}\sinh(x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x - (-e^{-x})}{2} = \cosh(x)$$

Likewise,

$$\frac{d}{dx}\cosh(x) = \sinh(x)$$

(Note that this is different from $\frac{d}{dx}\cos(x)$.)

Important identity:

$$\cosh^2(x) - \sinh^2(x) = 1$$

Proof:

$$\cosh^2(x) - \sinh^2(x) = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$\cosh^2(x) - \sinh^2(x) = \frac{1}{4}\left(e^{2x} + 2e^x e^{-x} + e^{-2x}\right) - \frac{1}{4}\left(e^{2x} - 2 + e^{-2x}\right) = \frac{1}{4}(2+2) = 1$$

Why are these functions called "hyperbolic"?
Let $u = \cosh(x)$ and $v = \sinh(x)$, then

$$u^2 - v^2 = 1$$

which is the equation of a hyperbola.

Regular trig functions are "circular" functions. If $u = \cos(x)$ and $v = \sin(x)$, then

$$u^2 + v^2 = 1$$

which is the equation of a circle.

Exam 1 Review

General Differentiation Formulas

$$
\begin{aligned}
(u+v)' &= u' + v' \\
(cu)' &= cu' \\
(uv)' &= u'v + uv' \quad \text{(product rule)} \\
\left(\frac{u}{v}\right)' &= \frac{u'v - uv'}{v^2} \quad \text{(quotient rule)} \\
\frac{d}{dx} f(u(x)) &= f'(u(x)) \cdot u'(x) \quad \text{(chain rule)}
\end{aligned}
$$

You can remember the quotient rule by rewriting

$$\left(\frac{u}{v}\right)' = (uv^{-1})'$$

and applying the product rule and chain rule.

Implicit differentiation

Let's say you want to find y' from an equation like

$$y^3 + 3xy^2 = 8$$

Instead of solving for y and then taking its derivative, just take $\frac{d}{dx}$ of the whole thing. In this example,

$$
\begin{aligned}
3y^2y' + 6xyy' + 3y^2 &= 0 \\
(3y^2 + 6xy)y' &= -3y^2 \\
y' &= \frac{-3y^2}{3y^2 + 6xy}
\end{aligned}
$$

Note that this formula for y' involves both x and y. Implicit differentiation can be very useful for taking the derivatives of inverse functions.

For instance,

$$y = \sin^{-1} x \Rightarrow \sin y = x$$

Implicit differentiation yields

$$(\cos y)y' = 1$$

and

$$y' = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - x^2}}$$

Specific differentiation formulas

You will be responsible for knowing formulas for the derivatives and how to deduce these formulas from previous information: x^n, $\sin^{-1} x$, $\tan^{-1} x$, $\sin x$, $\cos x$, $\tan x$, $\sec x$, e^x, $\ln x$.

For example, let's calculate $\frac{d}{dx}\sec x$:

$$\frac{d}{dx}\sec x = \frac{d}{dx}\frac{1}{\cos x} = \frac{-(-\sin x)}{\cos^2 x} = \tan x \sec x$$

You may be asked to find $\frac{d}{dx}\sin x$ or $\frac{d}{dx}\cos x$, using the following information:

$$\begin{aligned} \lim_{h\to 0}\frac{\sin(h)}{h} &= 1 \\ \lim_{h\to 0}\frac{\cos(h)-1}{h} &= 0 \end{aligned}$$

Remember the definition of the derivative:

$$\frac{d}{dx}f(x) = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

Tying up a loose end

How to find $\frac{d}{dx}x^r$, where r is a real (but not necessarily rational) number? All we have done so far is the case of rational numbers, using implicit differentiation. We can do this two ways:

1st method: base e

$$\begin{aligned} x &= e^{\ln x} \\ x^r &= \left(e^{\ln x}\right)^r = e^{r\ln x} \\ \frac{d}{dx}x^r &= \frac{d}{dx}e^{r\ln x} = e^{r\ln x}\frac{d}{dx}(r\ln x) = e^{r\ln x}\frac{r}{x} \\ \frac{d}{dx}x^r &= x^r\left(\frac{r}{x}\right) = rx^{r-1} \end{aligned}$$

2nd method: logarithmic differentiation

$$\begin{aligned} (\ln f)' &= \frac{f'}{f} \\ f &= x^r \\ \ln f &= r\ln x \\ (\ln f)' &= \frac{r}{x} \\ f' = f(\ln f)' &= x^r\left(\frac{r}{x}\right) = rx^{r-1} \end{aligned}$$

Finally, in the first lecture I promised you that you'd learn to differentiate *anything*— even something as complicated as

$$\frac{d}{dx} e^{x \tan^{-1} x}$$

So let's do it!

$$\frac{d}{dx} e^{uv} \quad = \quad e^{uv} \frac{d}{dx}(uv) \; = \; e^{uv}(u'v + uv')$$

Substituting,

$$\frac{d}{dx} e^{x \tan^{-1} x} \quad = \quad e^{x \tan^{-1} x} \left(\tan^{-1} x + x \left(\frac{1}{1+x^2} \right) \right)$$