18.01 Single Variable Calculus
Fall 2006

Lecture 5
Implicit Differentiation and Inverses

Implicit Differentiation

Example 1. $\frac{d}{dx}(x^a) = ax^{a-1}$.
We proved this by an explicit computation for $a = 0, 1, 2, \ldots$. From this, we also got the formula for $a = -1, -2, \ldots$. Let us try to extend this formula to cover rational numbers, as well:

$$a = \frac{m}{n}; \quad y = x^{\frac{m}{n}} \quad \text{where } m \text{ and } n \text{ are integers.}$$

We want to compute $\frac{dy}{dx}$. We can say $y^n = x^m$ so $ny^{n-1}\frac{dy}{dx} = mx^{m-1}$. Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{m}{n}\frac{x^{m-1}}{y^{n-1}}$$

We know that $y = x^{\left(\frac{m}{n}\right)}$ is a function of x.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{m}{n}\left(\frac{x^{m-1}}{y^{n-1}}\right) \\
&= \frac{m}{n}\left(\frac{x^{m-1}}{(x^{m/n})^{n-1}}\right) \\
&= \frac{m}{n}\frac{x^{m-1}}{x^{m(n-1)/n}} \\
&= \frac{m}{n}x^{(m-1)-\frac{m(n-1)}{n}} \\
&= \frac{m}{n}x^{\frac{n(m-1)-m(n-1)}{n}} \\
&= \frac{m}{n}x^{\frac{nm-n-nm+m}{n}} \\
&= \frac{m}{n}x^{\frac{m}{n}-\frac{n}{n}} \\
\text{So,} \quad \frac{dy}{dx} &= \frac{m}{n}x^{\frac{m}{n}-1}
\end{aligned}$$

This is the same answer as we were hoping to get!

Example 2. Equation of a circle with a radius of 1: $x^2 + y^2 = 1$ which we can write as $y^2 = 1 - x^2$. So $y = \pm\sqrt{1-x^2}$. Let us look at the positive case:

$$\begin{aligned}
y &= +\sqrt{1-x^2} = (1-x^2)^{\frac{1}{2}} \\
\frac{dy}{dx} &= \left(\frac{1}{2}\right)(1-x^2)^{\frac{-1}{2}}(-2x) = \frac{-x}{\sqrt{1-x^2}} = \frac{-x}{y}
\end{aligned}$$

Now, let's do the same thing, using *implicit* differentiation.

$$\begin{aligned} x^2+y^2 &= 1 \\ \frac{d}{dx}\left(x^2+y^2\right) &= \frac{d}{dx}(1)=0 \\ \frac{d}{dx}(x^2)+\frac{d}{dx}(y^2) &= 0 \end{aligned}$$

Applying chain rule in the second term,

$$\begin{aligned} 2x+2y\frac{dy}{dx} &= 0 \\ 2y\frac{dy}{dx} &= -2x \\ \frac{dy}{dx} &= \frac{-x}{y} \end{aligned}$$

Same answer!

Example 3. $y^3+xy^2+1=0$. In this case, it's not easy to solve for y as a function of x. Instead, we use implicit differentiation to find $\frac{dy}{dx}$.

$$3y^2\frac{dy}{dx}+y^2+2xy\frac{dy}{dx} = 0$$

We can now solve for $\frac{dy}{dx}$ in terms of y and x.

$$\begin{aligned} \frac{dy}{dx}(3y^2+2xy) &= -y^2 \\ \frac{dy}{dx} &= \frac{-y^2}{3y^2+2xy} \end{aligned}$$

Inverse Functions

If $y=f(x)$ and $g(y)=x$, we call g the *inverse function* of f, f^{-1}:

$$x=g(y)=f^{-1}(y)$$

Now, let us use implicit differentiation to find the derivative of the inverse function.

$$\begin{aligned} y &= f(x) \\ f^{-1}(y) &= x \\ \frac{d}{dx}(f^{-1}(y)) &= \frac{d}{dx}(x)=1 \end{aligned}$$

By the chain rule:

$$\frac{d}{dy}(f^{-1}(y))\frac{dy}{dx} = 1$$

and

$$\frac{d}{dy}(f^{-1}(y)) = \frac{1}{\frac{dy}{dx}}$$

So, implicit differentiation makes it possible to find the derivative of the inverse function.

Example. $y = \arctan(x)$

$$\begin{aligned}
\tan y &= x \\
\frac{d}{dx}[\tan(y)] &= \frac{dx}{dx} = 1 \\
\frac{d}{dy}[\tan(y)]\frac{dy}{dx} &= 1 \\
\left(\frac{1}{\cos^2(y)}\right)\frac{dy}{dx} &= 1 \\
\frac{dy}{dx} &= \cos^2(y) = \cos^2(\arctan(x))
\end{aligned}$$

This form is messy. Let us use some geometry to simplify it.

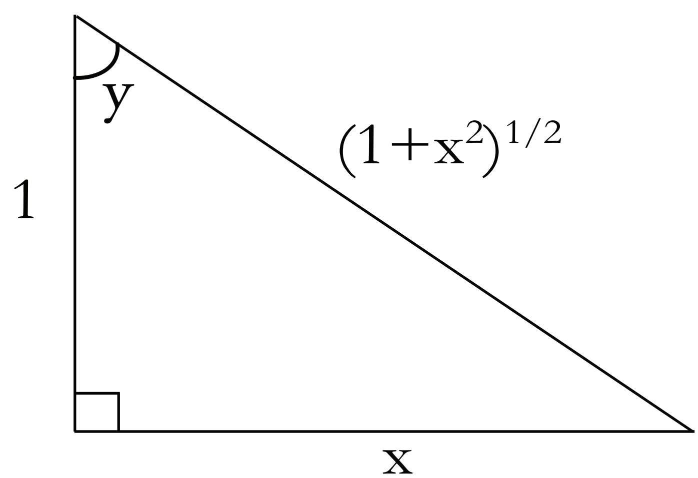

Figure 1: Triangle with angles and lengths corresponding to those in the example illustrating differentiation using the inverse function *arctan*

In this triangle, $\tan(y) = x$ so

$$\arctan(x) = y$$

The Pythagorian theorem tells us the length of the hypotenuse:

$$h = \sqrt{1 + x^2}$$

From this, we can find

$$\cos(y) = \frac{1}{\sqrt{1 + x^2}}$$

From this, we get

$$\cos^2(y) = \left(\frac{1}{\sqrt{1 + x^2}}\right)^2 = \frac{1}{1 + x^2}$$

So,

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

In other words,

$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$$

Graphing an Inverse Function.

Suppose $y = f(x)$ and $g(y) = f^{-1}(y) = x$. To graph g and f together we need to write g as a function of the variable x. If $g(x) = y$, then $x = f(y)$, and what we have done is to trade the variables x and y. This is illustrated in Fig. 2

$f^{-1}(f(x)) = x$	$f^{-1} \circ f(x) = x$
$f(f^{-1}(x)) = x$	$f \circ f^{-1}(x) = x$

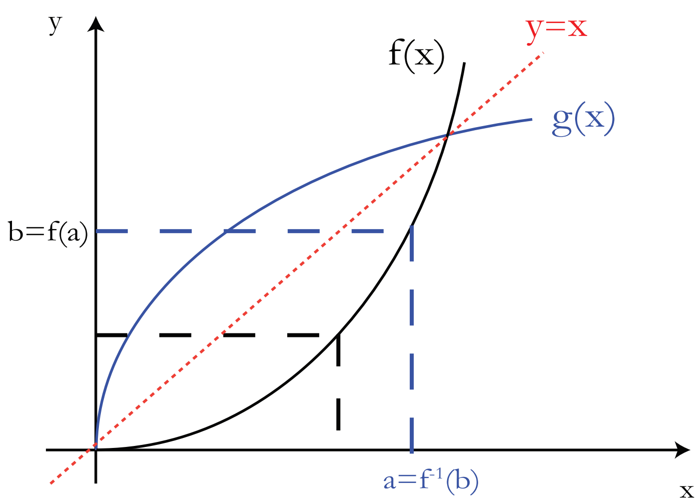

Figure 2: You can think about f^{-1} as the graph of f reflected about the line $y = x$