18.01 Single Variable Calculus
Fall 2006

Lecture 38: Final Review

Review: Differentiating and Integrating Series.

If $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad \int f(x)dx = C + \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}$$

Example 1: Normal (or Gaussian) Distribution.

$$\begin{aligned}
\int_0^x e^{-t^2} dt &= \int_0^x \left(1 - t^2 + \frac{(-t^2)^2}{2!} + \frac{(-t^2)^3}{3!} + \cdots\right) dt \\
&= \int_0^x \left(1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \frac{t^8}{4!} - \ldots\right) dt \\
&= x - \frac{x^3}{3} + \frac{1}{2!}\frac{x^5}{5} - \frac{1}{3!}\frac{x^7}{7} + \ldots
\end{aligned}$$

Even though $\int_0^x e^{-t^2}\, dt$ isn't an elementary function, we can still compute it. Elementary functions are still a little bit better, though. For example:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \implies \sin\frac{\pi}{2} = \frac{\pi}{2} - \frac{(\pi/2)^3}{3!} + \frac{(\pi/2)^5}{5!} - \cdots$$

But to compute $\sin(\pi/2)$ numerically is a waste of time. We know that the sum if something very simple, namely,

$$\sin\frac{\pi}{2} = 1$$

It's not obvious from the series expansion that $\sin x$ deals with angles. Series are sometimes complicated and unintuitive.

Nevertheless, we can read this formula backwards to find a formula for $\frac{\pi}{2}$. Start with $\sin\frac{\pi}{2} = 1$. Then,

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x\Big|_0^1 = \sin^{-1} 1 - \sin^{-1} 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

We want to find the series expansion for $(1-x^2)^{-1/2}$, but let's tackle a simpler case first:

$$\begin{aligned}
(1+u)^{-1/2} &= 1 + \left(-\frac{1}{2}\right)u + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{1\cdot 2}u^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{1\cdot 2\cdot 3}u^3 + \cdots \\
&= 1 - \frac{1}{2}u + \frac{1\cdot 3}{2\cdot 4}u^2 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}u^3 + \cdots
\end{aligned}$$

Notice the pattern: odd numbers go on the top, even numbers go on the bottom, and the signs alternate.

Now, let $u = -x^2$.

$$(1-x^2)^{-1/2} = 1 + \frac{1}{2}x^2 + \frac{1\cdot 3}{2\cdot 4}x^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6 + \cdots$$

$$\int (1-x^2)^{-1/2}dx = C + \left(x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7}\right) + \cdots$$

$$\frac{\pi}{2} = \int_0^1 (1-x^2)^{-1/2}dx = 1 + \frac{1}{2}\left(\frac{1}{3}\right) + \left(\frac{1\cdot 3}{2\cdot 4}\right)\left(\frac{1}{5}\right) + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\left(\frac{1}{7}\right) + \cdots$$

Here's a hard (optional) extra credit problem: why does this series converge? Hint: use L'Hôpital's rule to find out how quickly the terms decrease.

The Final Exam

Here's another attempt to clarify the concept of weighted averages.

Weighted Average

A weighted average of some function, f, is defined as:

$$\text{Average}(f) = \frac{\int_a^b w(x)f(x)\,dx}{\int_a^b w(x)\,dx}$$

Here, $\int_a^b w(x)\,dx$ is the total, and $w(x)$ is the weighting function.

Example: taken from a past problem set.
You get \$t if a certain particle decays in t seconds. How much should you pay to play? You were given that the likelihood that the particle has not decayed (the weighting function) is:

$$w(x) = e^{-kt}$$

Remember,

$$\int_0^\infty e^{-kt}\,dt = \frac{1}{k}$$

The payoff is

$$f(t) = t$$

The expected (or average) payoff is

$$\frac{\int_0^\infty f(t)w(t)\,dt}{\int_0^\infty w(t)\,dt} = \frac{\int_0^\infty te^{-kt}\,dt}{\int_0^\infty e^{-kt}\,dt}$$

$$= k\int_0^\infty te^{-kt}\,dt = \int_0^\infty (kt)e^{-kt}\,dt$$

Do the change of variable:

$$u = kt \quad \text{and} \quad du = k\,dt$$

$$\text{Average} = \int_0^\infty ue^{-u}\,\frac{du}{k}$$

On a previous problem set, you evaluated this using integration by parts: $\int_0^\infty ue^{-u}\,du = 1$.

$$\text{Average} = \int_0^\infty ue^{-u}\,\frac{du}{k} = \frac{1}{k}$$

On the problem set, we calculated the half-life (H) for Polonium^{120} was $(131)(24)(60)^2$ seconds. We also found that

$$k = \frac{\ln 2}{H}$$

Therefore, the expected payoff is

$$\frac{1}{k} = \frac{H}{\ln 2}$$

where H is the half-life of the particle in seconds.

Now, you're all probably wondering: who on earth bets on particle decays?

In truth, no one does. There is, however, a very similar problem that is useful in the real world. There is something called an annuity, which is basically a retirement pension. You can buy an annuity, and then get paid a certain amount every month once you retire. Once you die, the annuity payments stop.

You (and the people paying you) naturally care about how much money you can expect to get over the course of your retirement. In this case, $f(t) = t$ represents how much money you end up with, and $w(t) = e^{-kt}$ represents how likely your are to be alive after t years.

What if you want a 2-life annuity? Then, you need multiple integrals, which you will learn about in multivariable calculus (18.02).

Our first goal in this class was to be able to differentiate anything. In multivariable calculus, you will learn about another chain rule. That chain rule will unify the (single-variable) chain rule, the product rule, the quotient rule, and implicit differentiation.

You might say the multivariable chain rule is

One thing to rule them all
One thing to find them
One thing to bring them all
And in a matrix bind them.

(with apologies to JRR Tolkien).