18.01 Single Variable Calculus
Fall 2006

# Lecture 35: Improper Integrals

## Definition.

An *improper integral*, defined by

$$\int_a^\infty f(x)dx = \lim_{M\to\infty} \int_a^M f(x)dx$$

is said to converge if the limit exists (diverges if the limit does not exist).

**Example 1.** $\displaystyle\int_0^\infty e^{-kx}dx = 1/k \quad (k > 0)$

$$\int_0^M e^{-kx}dx = (-1/k)e^{-kx}\Big|_0^M = (1/k)(1 - e^{-kM})$$

Taking the limit as $M \to \infty$, we find $e^{-kM} \to 0$ and

$$\int_0^\infty e^{-kx}dx = 1/k$$

We rewrite this calculation more informally as follows,

$$\int_0^\infty e^{-kx}dx = (-1/k)e^{-kx}\Big|_0^\infty = (1/k)(1 - e^{-k\infty}) = 1/k \qquad \text{(since } k > 0\text{)}$$

Note that the integral over the infinite interval $\displaystyle\int_0^\infty e^{-kx}dx = 1/k$ has an easier formula than the corresponding finite integral $\displaystyle\int_0^M e^{-kx}dx = (1/k)(1 - e^{-kM})$. As a practical matter, for large $M$, the term $e^{-kM}$ is negligible, so even the simpler formula $1/k$ serves as a good approximation to the finite integral. Infinite integrals are often easier than finite ones, just as infinitesimals and derivatives are easier than difference quotients.

**Application:** Replace $x$ by $t = $ time in seconds in Example 1.
$R = $ rate of decay $=$ number of atoms that decay per second at time 0.
At later times $t > 0$ the decay rate is $Re^{-kt}$ (smaller by an exponential factor $e^{-kt}$)

Eventually (over time $0 \le t < \infty$) every atom decays. So the total number of atoms $N$ is calculated using the formula we found in Example 1,

$$N = \int_0^\infty Re^{-kt}dt = R/k$$

The half life $H$ of a radioactive element is the time $H$ at which the decay rate is half what it was at the start. Thus

$$e^{-kH} = 1/2 \quad \Longrightarrow \quad -kH = \ln(1/2) \quad \Longrightarrow \quad k = (\ln 2)/H$$

Hence
$$R = Nk = N(\ln 2)/H$$

Let us illustrate with Polonium 210, which has been in the news lately. The half life is 138 days or
$$H = (138\text{days})(24\text{hr/day})(60^2\text{sec/hr}) = (138)(24)(60)^2\text{seconds}$$

Using this value of $H$, we find that one gram of Polonium 210 emits $(1 \text{ gram})(6 \times 10^{23}/210$ atoms/gram$)(ln2)/H = 1.6610^{14}$ decays/sec $\approx 4500$ curies

At 5.3 MeV per decay, Polonium gives off 140 watts of radioactive energy per gram (white hot). Polonium emits alpha rays, which are blocked by skin but when ingested are 20 times more dangerous than gamma and X-rays. The lethal dose, when ingested, is about $10^{-7}$ grams.

**Example 2.** $\displaystyle\int_0^\infty dx/(1+x^2) = \pi/2$.
We calculate,
$$\int_0^M \frac{dx}{1+x^2} = \tan^{-1} x \Big|_0^M = \tan^{-1} M \to \pi/2$$

as $M \to \infty$. (If $\theta = \tan^{-1} M$ then $\theta \to \pi/2$ as $M \to \infty$. See Figures 1 and 2.)



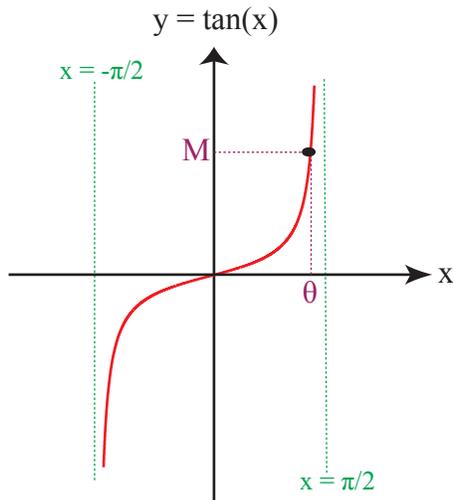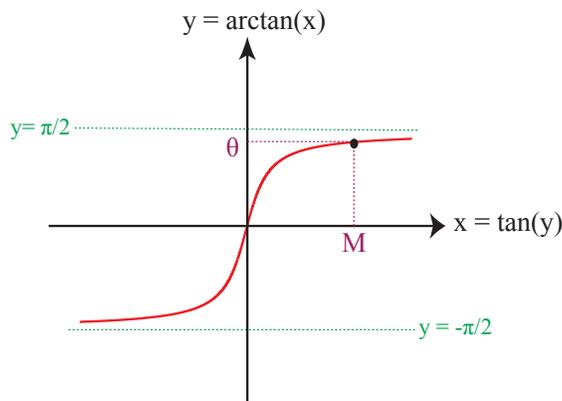Figure 1: Graph of the tangent function, $M = \tan\theta$.

Figure 2: Graph of the arctangent function, $\theta = \tan^{-1} M$.

**Example 3.** $\displaystyle\int_0^\infty e^{-x^2}\,dx = \sqrt{\pi}/2$

Recall that we already computed this improper integral (by computing a volume in two ways, slices and the method of shells). This shows vividly that a finite integral can be harder to understand than its infinite counterpart:

$$\int_0^M e^{-x^2}\,dx$$

can only evaluated numerically. It has no elementary formula. By contrast, we found an explicit formula when $M = \infty$.

**Example 4.** $\displaystyle\int_1^\infty dx/x$

$$\int_1^M dx/x = \ln x \Big|_1^M = \ln M - \ln 1 = \ln M \to \infty$$

as $M \to \infty$. This improper integral is infinite (called divergent or not convergent).

**Example 5.** $\displaystyle\int_1^\infty dx/x^p \quad (p > 1)$

$$\int_1^M dx/x^p = (1/(1-p))x^{1-p}\Big|_1^M = (1/(1-p))(M^{1-p} - 1) \to 1/(p-1)$$

as $M \to \infty$ because $1 - p < 0$. Thus, this integral is convergent.

**Example 6.** $\displaystyle\int_1^\infty dx/x^p \quad (0 < p < 1)$

This is very similar to the previous example, but diverges

$$\int_1^M dx/x^p = (1/(1-p))x^{1-p}\Big|_1^M = (1/(1-p))(M^{1-p} - 1) \to \infty$$

as $M \to \infty$ because $1 - p > 0$.

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# Determining Divergence and Convergence

To decide whether an integral converges or diverges, don't need to evaluate. Instead one can compare it to a simpler integral that can be evaluated.

**_The General Story for powers:_** $\displaystyle\int_1^\infty \frac{dx}{x^p}$

From Examples 4, 5 and 6 we know that this diverges (is infinite) for $0 < p \le 1$ and converges (is finite) for $p > 1$.

The comparison of integrals says that a larger function has a larger integral. If we restrict ourselves to nonnegative functions, then even when the region is unbounded, as in the case of an improper integral, the area under the graph of the larger function is more than the area under the graph of the smaller one. Consider $0 \le f(x) \le g(x)$ (as in Figure 3)
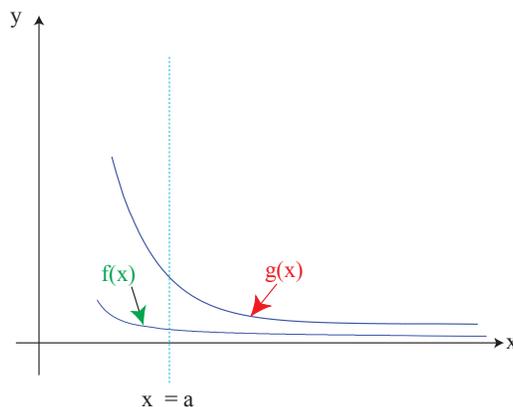


Figure 3: The area under $f(x)$ is less than the area under $g(x)$ for $a \le x < \infty$.

If $\displaystyle\int_a^\infty g(x)\,dx$ converges, then so does $\displaystyle\int_a^\infty f(x)\,dx$. (In other words, if the area under $g$ is finite, then the area under $f$, being smaller, must also be finite.)

If $\displaystyle\int_a^\infty f(x)\,dx$ diverges, then so does $\displaystyle\int_a^\infty g(x)\,dx$. (In other words, if the area under $f$ is infinite, then the area under $g$, being larger, must also be infinite.)

The way comparison is used is by replacing functions by simpler ones whose integrals we can calculate. You will have to decide whether you want to trap the function from above or below. This will depend on whether you are demonstrating that the integral is finite or infinite.

**Example 7.** $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x^3+1}}$ It is natural to try the comparison

$$\frac{1}{\sqrt{x^3+1}} \leq \frac{1}{x^{3/2}}$$

But the area under $x^{-3/2}$ on the interval $0 < x < \infty$,

$$\int_0^\infty \frac{dx}{x^{3/2}}$$

turns out to be infinite because of the infinite behavior as $x \to 0$. We can rescue this comparison by excluding an interval near 0.

$$\int_0^\infty \frac{dx}{\sqrt{x^3+1}} = \int_0^1 \frac{dx}{\sqrt{x^3+1}} + \int_1^\infty \frac{dx}{\sqrt{x^3+1}}$$

The integral on $0 < x < 1$ is a finite integral and the second integral now works well with comparison,

$$\int_1^\infty \frac{dx}{\sqrt{x^3+1}} \leq \int_1^\infty \frac{dx}{x^{3/2}} < \infty$$

because $3/2 > 1$.

**Example 8.** $\displaystyle\int_0^\infty e^{-x^3}\, dx$

For $x \geq 1$, $x^3 \geq x$, so

$$\int_1^\infty e^{-x^3}\, dx \leq \int_1^\infty e^{-x}dx = 1 < \infty$$

Thus the full integral from $0 \leq x < \infty$ of $e^{-x^3}$ converges as well. We can ignore the interval $0 \leq x \leq 1$ because it has finite length and $e^{-x^3}$ does not tend to infinity there.

**Limit comparison:**

Suppose that $0 \leq f(x)$ and $\displaystyle\lim_{x\to\infty} f(x)/g(x) \leq 1$. Then $f(x) \leq 2g(x)$ for $x \geq a$ (some large $a$).
Hence $\displaystyle\int_a^\infty f(x)\, dx \leq 2 \int_a^\infty g(x)\, dx$.

**Example 9.** $\displaystyle\int_0^\infty \frac{(x+10)\, dx}{x^2+1}$

The limiting behavior as $x \to \infty$ is

$$\frac{(x+10)dx}{x^2+1} \simeq \frac{x}{x^2} = \frac{1}{x}$$

Since $\displaystyle\int_1^\infty \frac{dx}{x} = \infty$, the integral $\displaystyle\int_0^\infty \frac{(x+10)\, dx}{x^2+1}$ also diverges.

**Example 10 (from PS8).** $\displaystyle\int_0^\infty x^n e^{-x} dx$

This converges. To carry out a convenient comparison requires some experience with growth rates of functions.

$x^n << e^x$ not enough. Instead use $x^n/e^{x/2} \to 0$ (true by L'Hop). It follows that

$$x^n << e^{x/2} \implies x^n e^{-x} << e^{x/2} e^{-x} = e^{-x/2}$$

Now by limit comparison, since $\displaystyle\int_0^\infty e^{-x/2} dx$ converges, so does our integral. You will deal with this integral on the problem set.

# Improper Integrals of the Second Type

$$\int_0^1 \frac{dx}{\sqrt{x}}$$

We know that $\dfrac{1}{\sqrt{x}} \to \infty$ as $x \to 0$.

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{a \to 0^+} \int_a^1 x^{-1/2} dx$$

$$\int_a^1 x^{-1/2} dx = 2x^{1/2}\Big|_a^1 = 2 - 2a^{1/2}$$

As $a \to 0$, $2a^{1/2} \to 0$. So,

$$\int_0^1 x^{-1/2}\, dx = 2$$

Similarly,

$$\int_0^1 x^{-p} dx = \frac{1}{-p+1}$$

for all $p < 1$.

For $p = \dfrac{1}{2}$,

$$\frac{1}{\left(-\dfrac{1}{2}\right)+1} = 2$$

However, for $p \geq 1$, the integral diverges.