18.01 Single Variable Calculus
Fall 2006

Lecture 31: Parametric Equations, Arclength, Surface Area

Arclength, continued

Example 1. Consider this parametric equation:

$$x = t^2 \quad y = t^3 \quad \text{for } 0 \le t \le 1$$

$$x^3 = (t^2)^3 = t^6; \quad y^2 = (t^3)^2 = t^6 \quad \Longrightarrow x^3 = y^2 \Longrightarrow y = x^{2/3} \qquad 0 \le x \le 1$$

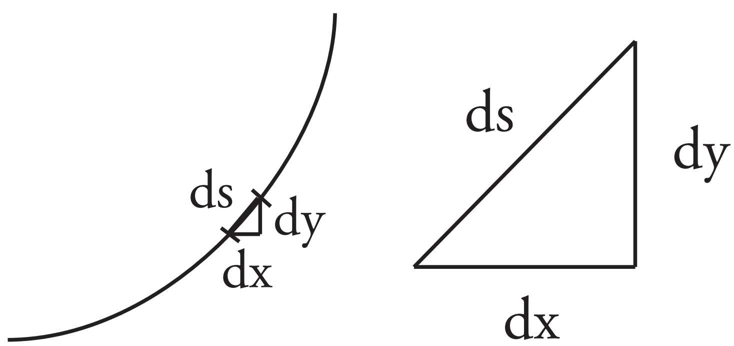

Figure 1: Infinitesimal Arclength.

$$(ds)^2 = (dx)^2 + (dy)^2$$

$$(ds)^2 = \underbrace{(2t\,dt)^2}_{(dx)^2} + \underbrace{(3t^2\,dt)^2}_{(dy)^2} = (4t^2 + 9t^4)(dt)^2$$

$$\text{Length} = \int_{t=0}^{t=1} ds = \int_0^1 \sqrt{4t^2 + 9t^4}dt = \int_0^1 t\sqrt{4 + 9t^2}dt$$

$$= \frac{(4 + 9t^2)^{3/2}}{27}\Big|_0^1 = \frac{1}{27}(13^{3/2} - 4^{3/2})$$

Even if you can't evaluate the integral analytically, you can always use numerical methods.

Surface Area (surfaces of revolution)

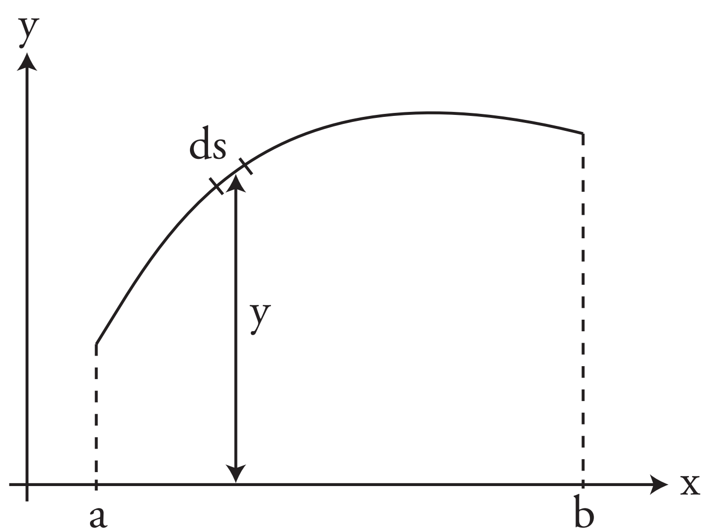

Figure 2: Calculating surface area

ds (the infinitesimal curve length in Figure 2) is revolved a distance $2\pi y$. The surface area of the thin strip of width ds is $2\pi y\, ds$.

Example 2. Revolve Example 1 ($x = t^2, y = t^3, 0 \le t \le 1$) around the x-axis. Refer to Figure 3.

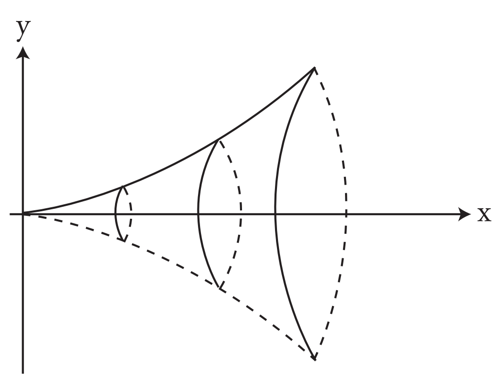

Figure 3: Curved surface of a trumpet.

$$\text{Area} = \int 2\pi y\, ds = \int_0^1 2\pi \underbrace{t^3}_{y} \underbrace{t\sqrt{4+9t^2}\, dt}_{ds} = 2\pi \int_0^1 t^4\sqrt{4+9t^2}\, dt$$

Now, we discuss the method used to evaluate

$$\int t^4(4+9t^2)^{1/2} dt$$

We're going to ignore the factor of 2π. You can reinsert it once you're done evaluating the integral. We use the trigonometric substitution

$$t = \frac{2}{3}\tan u; \quad dt = \frac{2}{3}\sec^2 u\, du; \quad \tan^2 u + 1 = \sec^2 u$$

Putting all of this together gives us:

$$\begin{aligned}
\int t^4(4+9t^2)^{1/2}\, dt &= \int \left(\frac{2}{3}\tan u\right)^4 \left(4 + 9\left(\frac{4}{9}\tan^2 u\right)\right)^{1/2} \left(\frac{2}{3}\sec^2 u\, du\right) \\
&= \left(\frac{2}{3}\right)^5 \int \tan^4 u (2\sec u)(\sec^2 u\, du)
\end{aligned}$$

This is a $\tan - \sec$ integral. It's doable, but it will take a long time for you to work the whole thing out. We're going to stop evaluating it here.

Example 3 Let's use what we've learned to find the surface area of the unit sphere (see Figure 4).

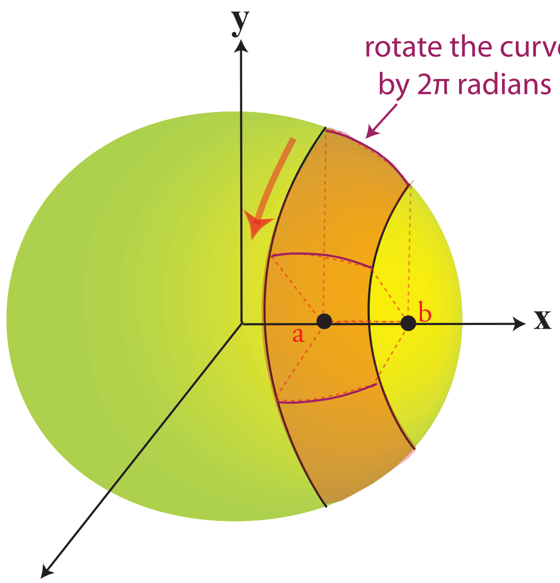

Figure 4: Slice of spherical surface (orange peel, only, not the insides).

For the top half of the sphere,

$$y = \sqrt{1 - x^2}$$

We want to find the area of the spherical slice between $x = a$ and $x = b$. A spherical slice has area

$$A = \int_{x=a}^{x=b} 2\pi y \, ds$$

From last time,

$$ds = \frac{dx}{\sqrt{1 - x^2}}$$

Plugging that in yields a remarkably simple formula for A:

$$A = \int_a^b 2\pi \sqrt{1 - x^2} \frac{dx}{\sqrt{1 - x^2}} = \int_a^b 2\pi \, dx$$

$$= 2\pi(b - a)$$

Special Cases

For a whole sphere, $a = -1$, and $b = 1$.

$$2\pi(1 - (-1)) = 4\pi$$

is the surface area of a unit sphere.

For a half sphere, $a = 0$ and $b = 1$.

$$2\pi(1 - 0) = 2\pi$$