18.01 Single Variable Calculus
Fall 2006

Lecture 30: Integration by Parts, Reduction Formulae

Integration by Parts

Remember the product rule:

$$(uv)' = u'v + uv'$$

We can rewrite that as

$$uv' = (uv)' - u'v$$

Integrate this to get the formula for integration by parts:

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example 1. $\int \tan^{-1} x\,dx$.
At first, it's not clear how integration by parts helps. Write

$$\int \tan^{-1} x\,dx = \int \tan^{-1} x(1\cdot dx) = \int uv'\,dx$$

with

$$u = \tan^{-1} x \quad \text{and} \quad v' = 1.$$

Therefore,

$$v = x \quad \text{and} \quad u' = \frac{1}{1+x^2}$$

Plug all of these into the formula for integration by parts to get:

$$\int \tan^{-1} x\,dx = \int uv'\,dx = (\tan^{-1} x)x - \int \frac{1}{1+x^2}(x)dx$$

$$= x\tan^{-1} x - \frac{1}{2}\ln|1+x^2| + c$$

Alternative Approach to Integration by Parts

As above, the product rule:

$$(uv)' = u'v + uv'$$

can be rewritten as

$$uv' = (uv)' - u'v$$

This time, let's take the *definite* integral:

$$\int_a^b uv'\,dx = \int_a^b (uv)'\,dx - \int_a^b u'v\,dx$$

By the fundamental theorem of calculus, we can say

$$\int_a^b uv'\,dx = uv\Big|_a^b - \int_a^b u'v\,dx$$

Another notation in the indefinite case is

$$\int u\,dv = uv - \int v\,du$$

This is the same because

$$dv = v'\,dx \implies uv'\,dx = u\,dv \quad \text{and} \quad du = u'\,dx \implies u'v\,dx = vu'\,dx = v\,du$$

Example 2. $\int (\ln x) dx$

$$u = \ln x;\ du = \frac{1}{x}dx \quad \text{and} \quad dv = dx;\ v = x$$

$$\int (\ln x)dx = x\ln x - \int x\left(\frac{1}{x}\right)dx = x\ln x - \int dx = x\ln x - x + c$$

We can also use "advanced guessing" to solve this problem. We know that the derivative of *something* equals $\ln x$:

$$\frac{d}{dx}(??) = \ln x$$

Let's try

$$\frac{d}{dx}(x\ln x) = \ln x + x\cdot\frac{1}{x} = \ln x + 1$$

That's almost it, but not quite. Let's repair this guess to get:

$$\frac{d}{dx}\left(x\ln x - x\right) = \ln x + 1 - 1 = \ln x$$

Reduction Formulas (Recurrence Formulas)

Example 3. $\int (\ln x)^n\,dx$

Let's try:

$$u = (\ln x)^n \implies u' = n(\ln x)^{n-1}\left(\frac{1}{x}\right)$$

$$v' = dx;\ v = x$$

Plugging these into the formula for integration by parts gives us:

$$\int (\ln x)^n dx = x(\ln x)^n - \int n(\ln x)^{n-1} \cancelto{1}{x\left(\frac{1}{x}\right)} dx$$

Keep repeating integration by parts to get the full formula: $n \to (n-1) \to (n-2) \to (n-3) \to$ etc

Example 4. $\int x^n e^x\,dx$ Let's try:

$$u = x^n \implies u' = nx^{n-1};\quad v' = e^x \implies v = e^x$$

Putting these into the integration by parts formula gives us:

$$\int x^n e^x \, dx = x^n e^x - \int n x^{n-1} e^x \, dx$$

Repeat, going from $n \to (n-1) \to (n-2) \to$ etc.

Bad news: If you change the integrals just a little bit, they become impossible to evaluate:

$$\int \left(\tan^{-1} x\right)^2 dx = \text{impossible}$$

$$\int \frac{e^x}{x} \, dx = \text{also impossible}$$

Good news: When you can't evaluate an integral, then

$$\int_1^2 \frac{e^x}{x} dx$$

is an *answer*, not a question. This *is* the solution– you don't have to integrate it!

The most important thing is setting up the integral! (Once you've done that, you can always evaluate it numerically on a computer.) So, why bother to evaluate integrals by hand, then? Because you often get families of related integrals, such as

$$F(a) = \int_1^\infty \frac{e^x}{x^a} dx$$

where you want to find how the answer depends on, say, a.

Arc Length

This is very useful to know for 18.02 (multi-variable calculus).

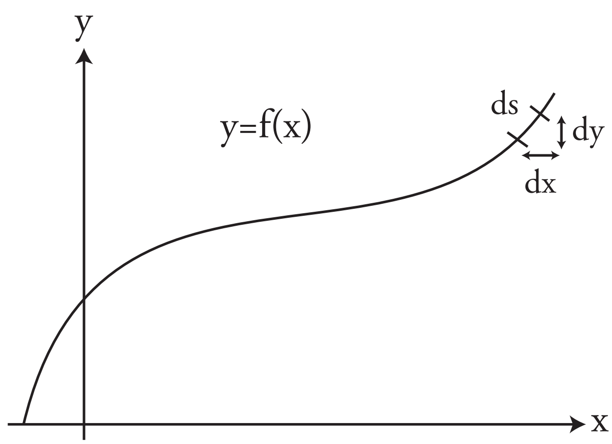

Figure 1: Infinitesimal Arc Length ds

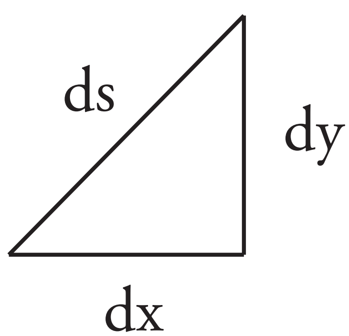

Figure 2: Zoom in on Figure 1 to see an approximate right triangle.

In Figures 1 and 2, s denotes arc length and ds = the infinitesimal of arc length.

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + (dy/dx)^2}\,dx$$

Integrating with respect to ds finds the length of a curve between two points (see Figure 3). To find the length of the curve between P_0 and P_1, evaluate:

$$\int_{P_0}^{P_1} ds$$

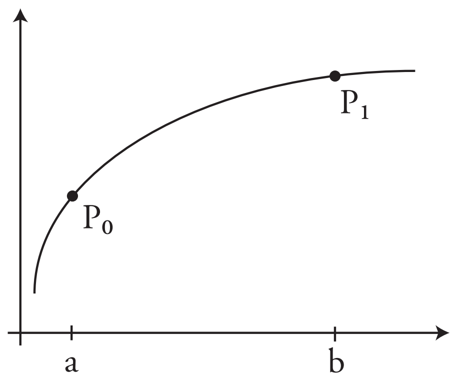

Figure 3: Find length of curve between P_0 and P_1.

We want to integrate with respect to x, not s, so we do the same algebra as above to find ds in terms of dx.

$$\frac{(ds)^2}{(dx)^2} = \frac{(dx)^2}{(dx)^2} + \frac{(dy)^2}{(dx)^2} = 1 + \left(\frac{dy}{dx}\right)^2$$

Therefore,

$$\int_{P_0}^{P_1} ds = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Example 5: The Circle. $x^2 + y^2 = 1$ (see Figure 4).

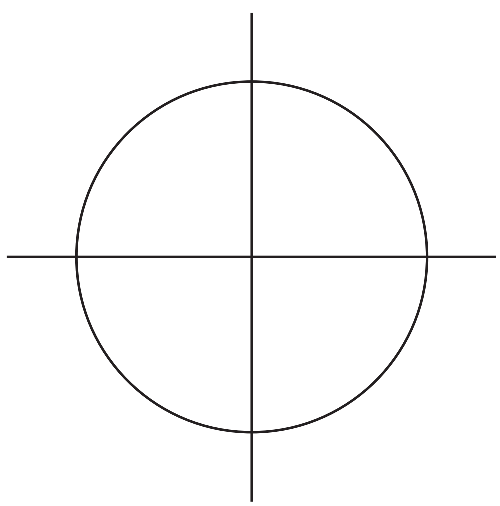

Figure 4: The circle in Example 1.

We want to find the length of the arc in Figure 5:

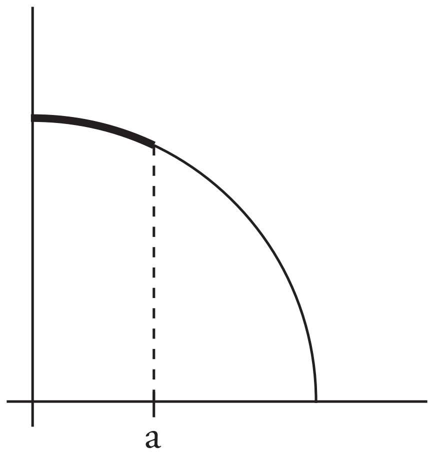

Figure 5: Arc length to be evaluated.

$$y = \sqrt{1-x^2}$$

$$\frac{dy}{dx} = \frac{-2x}{\sqrt{1-x^2}}\left(\frac{1}{2}\right) = \frac{-x}{\sqrt{1-x^2}}$$

$$ds = \sqrt{1+\left(\frac{-x}{\sqrt{1-x^2}}\right)^2}\,dx$$

$$1+\left(\frac{-x}{\sqrt{1-x^2}}\right)^2 = 1+\frac{x^2}{1-x^2} = \frac{1-x^2+x^2}{1-x^2} = \frac{1}{1-x^2}$$

$$ds = \sqrt{\frac{1}{1-x^2}}\,dx$$

$$s = \int_0^a \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x\,\Big|_0^a = \sin^{-1} a - \sin^{-1} 0 = \sin^{-1} a$$

$$\sin s = a$$

This is illustrated in Figure 6.

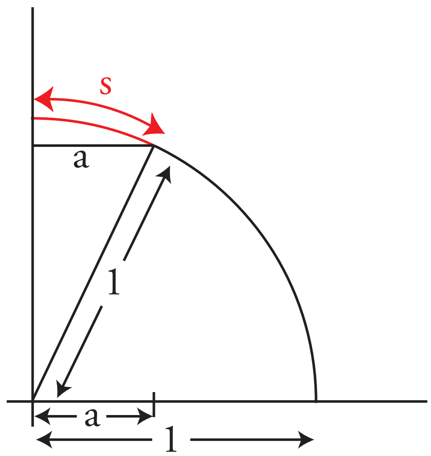

Figure 6: s = angle in radians.

Parametric Equations

Example 6.

$$x = a\cos t$$

$$y = a\sin t$$

Ask yourself: what's constant? What's varying? Here, t is variable and a is constant.
Is there a relationship between x and y? Yes:

$$x^2 + y^2 = a^2\cos^2 t + a^2\sin^2 t = a^2$$

Extra information (besides the circle):
At $t = 0$,

$$x = a\cos 0 = a \quad \text{and} \quad y = a\sin 0 = 0$$

At $t = \frac{\pi}{2}$,

$$x = a\cos\frac{\pi}{2} = 0 \quad \text{and} \quad y = a\sin\frac{\pi}{2} = a$$

Thus, for $0 \leq t \leq \pi/2$, a quarter circle is traced counter-clockwise (Figure 7).

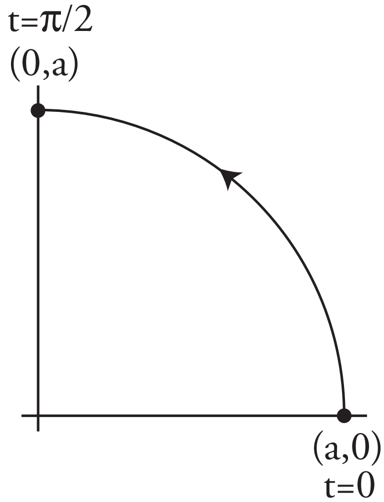

Figure 7: Example 6. $x = a\cos t,\ y = a\sin t$; the particle is moving counterclockwise.

Example 7: The Ellipse See Figure 8.

$$x = 2\sin t; \quad y = \cos t$$

$$\frac{x^2}{4} + y^2 = 1(\implies (2\sin t)^2/4 + (\cos t)^2 = \sin^2 t + \cos^2 t = 1)$$

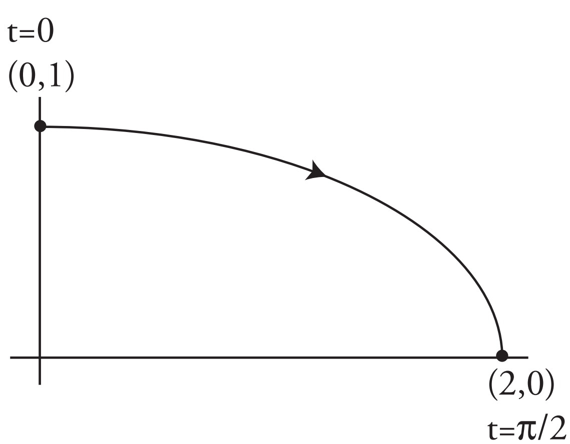

Figure 8: Ellipse: $x = 2\sin t,\ y = \cos t$ (traced clockwise).

Arclength ds for Example 6.

$$dx = -a\sin t\,dt, \quad dy = a\cos t\,dt$$

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{(-a\sin t\,dt)^2 + (a\cos t\,dt)^2} = \sqrt{(a\sin t)^2 + (a\cos t)^2}\,dt = a\,dt$$