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18.01 Single Variable Calculus
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Lecture 26: Trigonometric Integrals and Substitution

Trigonometric Integrals

How do you integrate an expression like $\int \sin^n x \cos^m x \, dx$? ($n = 0, 1, 2...$ and $m = 0, 1, 2, \ldots$)

We already know that:

$$\int \sin x \, dx = -\cos x + c \quad \text{and} \quad \int \cos x \, dx = \sin x + c$$

Method A

Suppose either n or m is odd.

Example 1. $\int \sin^3 x \cos^2 x \, dx$.
Our strategy is to use $\sin^2 x + \cos^2 x = 1$ to rewrite our integral in the form:

$$\int \sin^3 x \cos^2 x \, dx = \int f(\cos x) \, \sin x \, dx$$

Indeed,

$$\int \sin^3 x \cos^2 x \, dx = \int \sin^2 x \cos^2 x \, \sin x \, dx = \int (1 - \cos^2 x) \cos^2 x \, \sin x \, dx$$

Next, use the substitution

$$u = \cos x \quad \text{and} \quad du = -\sin x \, dx$$

Then,

$$\int (1 - \cos^2 x) \, \cos^2 x \, \sin x \, dx = \int (1 - u^2) u^2 (-du)$$

$$= \int (-u^2 + u^4) du = -\frac{1}{3} u^3 + \frac{1}{5} u^5 + c = -\frac{1}{3} \cos^3 u + \frac{1}{5} \cos^5 x + c$$

Example 2.

$$\int \cos^3 x \, dx = \int f(\sin x) \, \cos x \, dx = \int (1 - \sin^2 x) \, \cos x \, dx$$

Again, use a substitution, namely

$$u = \sin x \quad \text{and} \quad du = \cos x \, dx$$

$$\int \cos^3 x \, dx = \int (1 - u^2) du = u - \frac{u^3}{3} + c = \sin x - \frac{\sin^3 x}{3} + c$$

Method B

This method requires *both* m and n to be even. It requires double-angle formulae such as

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

(Recall that $\cos 2x = \cos^2 x - \sin^2 x = \cos^2 x - (1-\sin^2 x) = 2\cos^2 x - 1$)
Integrating gets us

$$\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}dx = \frac{x}{2} + \frac{\sin(2x)}{4} + c$$

We follow a similar process for integrating $\sin^2 x$.

$$\sin^2 x = \frac{1-\cos(2x)}{2}$$

$$\int \sin^2 x\,dx = \int \frac{1-\cos(2x)}{2}dx = \frac{x}{2} - \frac{\sin(2x)}{4} + c$$

The full strategy for these types of problems is to keep applying Method B until you can apply Method A (when one of m or n is odd).

Example 3. $\int \sin^2 x \cos^2 x\,dx$.
Applying Method B twice yields

$$\int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)dx = \int \left(\frac{1}{4} - \frac{1}{4}\cos^2 2x\right)dx$$

$$= \int \left(\frac{1}{4} - \frac{1}{8}(1+\cos 4x)\right)dx = \frac{1}{8}x - \frac{1}{32}\sin 4x + c$$

There is a shortcut for Example 3. Because $\sin 2x = 2\sin x\cos x$,

$$\int \sin^2 x\cos^2 x\,dx = \int \left(\frac{1}{2}\sin 2x\right)^2 dx = \frac{1}{4}\int \frac{1-\cos 4x}{2}dx = \text{ same as above}$$

The next family of trig integrals, which we'll start today, but will not finish is:

$$\int \sec^n x\,\tan^m x\,dx \quad \text{where } n = 0,1,2,\ldots \text{ and } m = 0,1,2,\ldots$$

Remember that

$$\sec^2 x = 1 + \tan^2 x$$

which we double check by writing

$$\frac{1}{\cos^2 x} = 1 + \frac{\sin^2 x}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^3 x}$$

$$\boxed{\int \sec^2 x\,dx = \tan x + c \qquad \int \sec x\,\tan x\,dx = \sec x + c}$$

To calculate the integral of $\tan x$, write

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx$$

Let $u = \cos x$ and $du = -\sin x \, dx$, then

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = \int -\frac{du}{u} = -\ln(u) + c$$

$$\boxed{\int \tan x \, dx = -\ln(\cos x) + c}$$

(We'll figure out what $\int \sec x \, dx$ is later.)

Now, let's see what happens when you have an even power of secant. (The case n even.)

$$\int \sec^4 x \, dx = \int f(\tan x) \, \sec^2 x \, dx = \int (1 + \tan^2 x) \, \sec^2 x \, dx$$

Make the following substitution:

$$u = \tan x$$

and

$$du = \sec^2 x \, dx$$

$$\int \sec^4 x \, dx = \int (1 + u^2) du = u + \frac{u^3}{3} + c = \tan x + \frac{\tan^3 x}{3} + c$$

What happens when you have a odd power of tan? (The case m odd.)

$$\int \tan^3 x \, \sec x \, dx = \int f(\sec x) \, d(\sec x)$$

$$= \int (\sec^2 x - 1) \, \sec x \, \tan x \, dx$$

(Remember that $\sec^2 x - 1 = \tan^2 x$.)

Use substitution:

$$u = \sec x$$

and

$$du = \sec x \, \tan x \, dx$$

Then,

$$\int \tan^3 x \, \sec x \, dx = \int (u^2 - 1) du = \frac{u^3}{3} - u + c = \frac{\sec^3 x}{3} - \sec x + c$$

We carry out one final case: $n = 1, m = 0$

$$\int \sec x \, dx = \ln(\tan x + \sec x) + c$$

We get the answer by "advanced guessing," i.e., "knowing the answer ahead of time."

$$\int \sec x\,dx = \sec x\left(\frac{\sec x + \tan x}{\sec x + \tan x}\right)dx = \int \frac{\sec^2 x + \sec x \tan x}{\tan x + \sec x}dx$$

Make the following substitutions:

$$u = \tan x + \sec x$$

and

$$du = (\sec^2 x + \sec x \tan x)\,dx$$

This gives

$$\int \sec x\,dx = \int \frac{du}{u} = \ln(u) + c = \ln(\tan x + \sec x) + c$$

Cases like $n = 3, m = 0$ or more generally n odd and m even are more complicated and will be discussed later.

Trigonometric Substitution

Knowing how to evaluate all of these trigonometric integrals turns out to be useful for evaluating integrals involving square roots.

Example 4. $y = \sqrt{a^2 - x^2}$

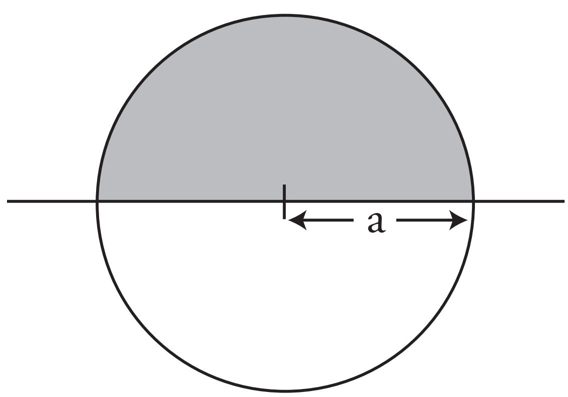

Figure 1: Graph of the circle $x^2 + y^2 = a^2$.

We already know that the area of the top half of the disk is

$$\int_{-a}^{a} \sqrt{a^2 - x^2}\,dx = \frac{\pi a^2}{2}$$

What if we want to find this area?

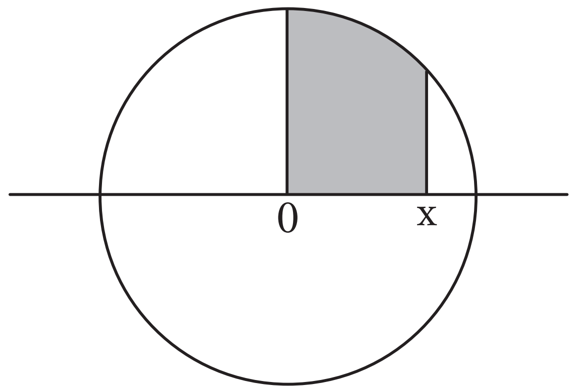

Figure 2: Area to be evaluated is shaded.

To do so, you need to evaluate this integral:

$$\int_{t=0}^{t=x} \sqrt{a^2 - t^2}\, dt$$

Let $t = a \sin u$ and $dt = a \cos u\, du$. (Remember to change the limits of integration when you do a change of variables.)
Then,

$$a^2 - t^2 = a^2 - a^2 \sin^2 u = a^2 \cos^2 u; \quad \sqrt{a^2 - t^2} = a \cos u$$

Plugging this into the integral gives us

$$\int_0^x \sqrt{a^2 - t^2}\, dt = \int (a \cos u)\, a \cos u\, du = a^2 \int_{u=0}^{u=\sin^{-1}(x/a)} \cos^2 u\, du$$

Here's how we calculated the new limits of integration:

$$\begin{aligned} t &= 0 \implies a \sin u = 0 \implies u = 0 \\ t &= x \implies a \sin u = x \implies u = \sin^{-1}(x/a) \end{aligned}$$

$$\int_0^x \sqrt{a^2 - t^2}\, dt = a^2 \int_0^{\sin^{-1}(x/a)} \cos^2 u\, du = a^2 \left(\frac{u}{2} + \frac{\sin 2u}{4} \right) \Big|_0^{\sin^{-1}(x/a)}$$

$$= \frac{a^2 \sin^{-1}(x/a)}{2} + \left(\frac{a^2}{4} \right) \left(2 \sin(\sin^{-1}(x/a)) \cos(\sin^{-1}(x/a)) \right)$$

(Remember, $\sin 2u = 2 \sin u\, \cos u$.)

We'll pick up from here next lecture (Lecture 28 since Lecture 27 is Exam 3).