18.01 Single Variable Calculus
Fall 2006

Lecture 23: Work, Average Value, Probability

Application of Integration to Average Value

You already know how to take the average of a set of discrete numbers:

$$\frac{a_1 + a_2}{2} \text{ or } \frac{a_1 + a_2 + a_3}{3}$$

Now, we want to find the average of a continuum.

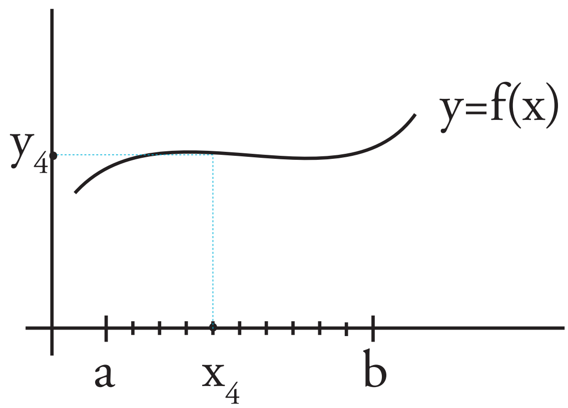

Figure 1: Discrete approximation to $y = f(x)$ on $a \leq x \leq b$.

$$\text{Average } \approx \frac{y_1 + y_2 + ... + y_n}{n}$$

where

$$a = x_0 < x_1 < \cdots x_n = b$$
$$y_0 = f(x_0),\, y_1 = f(x_1),\, \ldots y_n = f(x_n)$$

and

$$n(\Delta x) = b - a \qquad \Longleftrightarrow \qquad \Delta x = \frac{b-a}{n}$$

and

The limit of the Riemann Sums is

$$\lim_{n\to\infty} (y_1 + \cdots + y_n)\frac{b-a}{n} = \int_a^b f(x)\,dx$$

Divide by $b - a$ to get the continuous average

$$\lim_{n\to\infty} \frac{y_1 + \cdots + y_n}{n} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

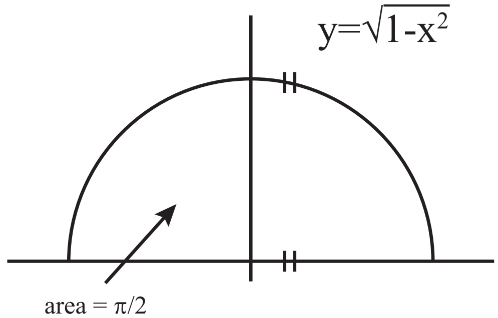

Figure 2: Average height of the semicircle.

Example 1. Find the average of $y = \sqrt{1-x^2}$ on the interval $-1 \le x \le 1$. (See Figure 2)

$$\text{Average height} = \frac{1}{2}\int_{-1}^{1}\sqrt{1-x^2}dx = \frac{1}{2}\left(\frac{\pi}{2}\right) = \frac{\pi}{4}$$

Example 2. The average of a constant is the same constant

$$\frac{1}{b-a}\int_a^b 53\,dx = 53$$

Example 3. Find the average height y on a semicircle, with respect to *arclength*. (Use $d\theta$ not dx. See Figure 3)

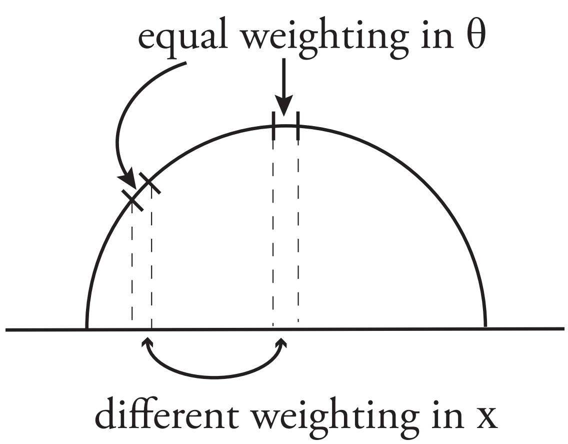

Figure 3: Different weighted averages.

$$y = \sin\theta$$
$$\text{Average } = \frac{1}{\pi}\int_0^{\pi} \sin\theta\,d\theta = \frac{1}{\pi}(-\cos\theta)\Big|_0^{\pi} = \frac{1}{\pi}\left(-\cos\pi - (-\cos 0)\right) = \frac{2}{\pi}$$

Example 4. Find the average temperature of water in the witches cauldron from last lecture. (See Figure 4).

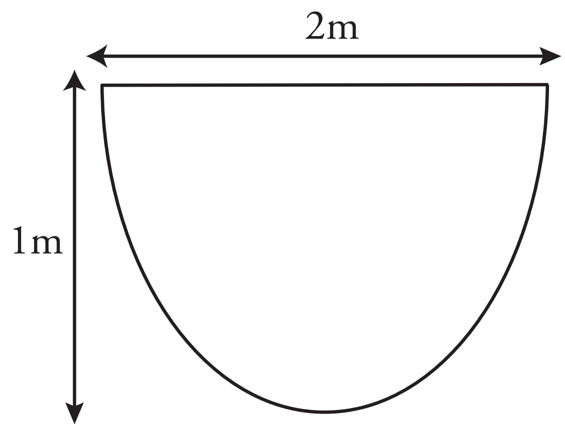

Figure 4: $y = x^2$, rotated about the y-axis.

First, recall how to find the volume of the solid of revolution by disks.

$$V = \int_0^1 (\pi x^2)\,dy = \int_0^1 \pi y\,dy = \frac{\pi y^2}{2}\Big|_0^1 = \frac{\pi}{2}$$

Recall that $T(y) = 100 - 30y$ and $(T(0) = 100^o; T(1) = 70^o)$. The average temperature per unit volume is computed by giving an importance or "weighting" $w(y) = \pi y$ to the disk at height y.

$$\frac{\int_0^1 T(y)w(y)\,dy}{\int_0^1 w(y)\,dy}$$

The numerator is

$$\int_0^1 T\pi y\,dy = \pi\int_0^1 (100-30y)y dy = \pi(500y^2 - 10y^3)\Big|_0^1 = 40\pi$$

Thus the average temperature is:

$$\frac{40\pi}{\pi/2} = 80^o C$$

Compare this with the average taken with respect to height y:

$$\frac{1}{1}\int_0^1 T\,dy = \int_0^1 (100-30y)dy = (100y - 15y^2)\Big|_0^1 = 85^o C$$

T is linear. Largest $T = 100^o C$, smallest $T = 70^o C$, and the average of the two is

$$\frac{70+100}{2} = 85$$

The answer 85^o is consistent with the ordinary average. The weighted average (integration with respect to $\pi y\,dy$) is lower (80^o) because there is more water at cooler temperatures in the upper parts of the cauldron.

Dart board, revisited

Last time, we said that the accuracy of your aim at a dart board follows a "normal distribution":

$$ce^{-r^2}$$

Now, let's pretend someone – say, your little brother – foolishly decides to stand close to the dart board. What is the chance that he'll get hit by a stray dart?

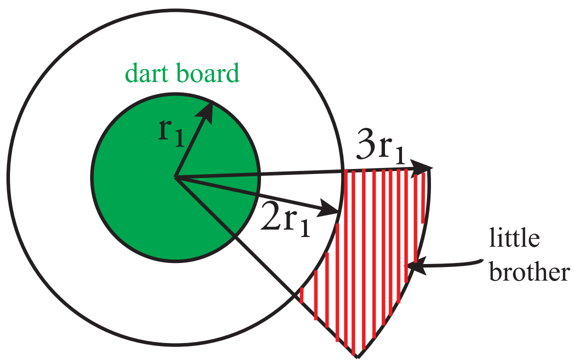

Figure 5: Shaded section is $2r_i < r < 3r_1$ between 3 and 5 o'clock.

To make our calculations easier, let's approximate your brother as a sector (the shaded region in Fig. 5). Your brother doesn't quite stand in front of the dart board. Let us say he stands at a distance r from the center where $2r_1 < r < 3r_1$ and r_1 is the radius of the dart board. Note that your brother doesn't surround the dart board. Let us say he covers the region between 3 o'clock and 5 o'clock, or $\frac{1}{6}$ of a ring.

Remember that

$$\boxed{\text{probability} = \frac{\text{part}}{\text{whole}}}$$

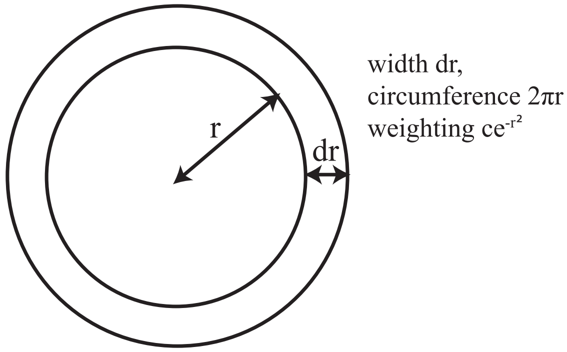

Figure 6: Integrating over rings.

The ring has weight $\left(ce^{-r^2}\right)(2\pi r)(dr)$ (see Figure 6). The probability of a dart hitting your brother is:

$$\frac{\frac{1}{6}\int_{2r_1}^{3r_1} ce^{-r^2} 2\pi r\,dr}{\int_0^\infty ce^{-r^2} 2\pi r\,dr}$$

Recall that $\dfrac{1}{6} = \dfrac{5-3}{12}$ is our approximation to the portion of the circumference where the little brother stands. (Note: $e^{-r^2} = e^{(-r^2)}$ not $(e^{-r})^2$)

$$\int_a^b re^{-r^2} dr = -\frac{1}{2}e^{-r^2}\Big|_a^b = -\frac{1}{2}e^{-b^2} + \frac{1}{2}e^{-a^2} \qquad \left(\frac{d}{dr}e^{-r^2} = -2re^{-r^2}\right)$$

Denominator:

$$\int_0^\infty e^{-r^2} r dr = -\frac{1}{2}e^{-r^2}\Big|_0^{R\to\infty} = -\frac{1}{2}e^{-R^2} + \frac{1}{2}e^{-0^2} = \frac{1}{2}$$

(Note that $e^{-R^2} \to 0$ as $R \to \infty$.)

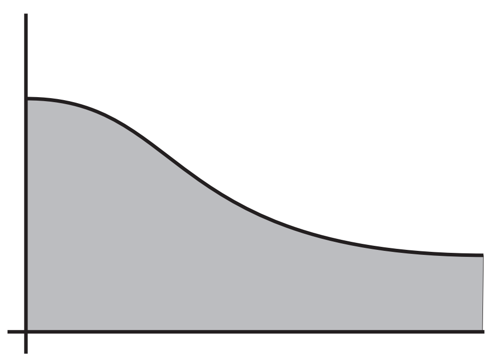

Figure 7: Normal Distribution.

$$\text{Probability} = \frac{\frac{1}{6}\int_{2r_1}^{3r_1} ce^{-r^2} 2\pi r\,dr}{\int_0^\infty ce^{-r^2} 2\pi r\,dr} = \frac{\frac{1}{6}\int_{2r_1}^{3r_1} e^{-r^2} r\,dr}{\int_0^\infty e^{-r^2} r\,dr} = \frac{1}{3}\int_{2r_1}^{3r_1} e^{-r^2} r\,dr = \frac{-e^{-r^2}}{6}\Big|_{2r_1}^{3r_1}$$

$$\text{Probability} = \frac{-e^{-9r_1^2} + e^{-4r_1^2}}{6}$$

Let's assume that the person throwing the darts hits the dartboard $0 \leq r \leq r_1$ about half the time. (Based on personal experience with 7-year-olds, this is realistic.)

$$P(0 \leq r \leq r_1) = \frac{1}{2} = \int_0^{r_1} 2e^{-r^2} r dr = -e^{-r_1^2} + 1 \implies e^{-{r_1}^2} = \frac{1}{2}$$

$$e^{-r_1^2} = \frac{1}{2}$$

$$e^{-9r_1^2} = \left(e^{-r_1^2}\right)^9 = \left(\frac{1}{2}\right)^9 \approx 0$$

$$e^{-4r_1^2} = \left(e^{-r_1^2}\right)^4 = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$$

So, the probability that a stray dart will strike your little brother is

$$\left(\frac{1}{16}\right)\left(\frac{1}{6}\right) \approx \frac{1}{100}$$

In other words, there's about a 1% chance he'll get hit with each dart thrown.

Volume by Slices: An Important Example

Compute $Q = \int_{-\infty}^{\infty} e^{-x^2}\,dx$

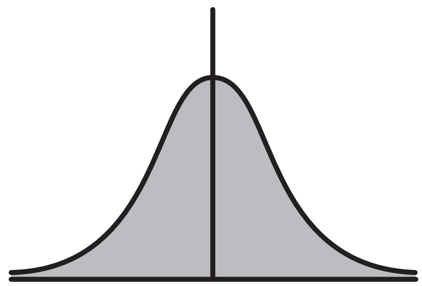

Figure 8: $Q =$ Area under curve $e^{(-x^2)}$.

This is one of the most important integrals in all of calculus. It is especially important in probability and statistics. It's an improper integral, but don't let those ∞'s scare you. In this integral, they're actually easier to work with than finite numbers would be.

To find Q, we will first find a volume of revolution, namely,

$$V = \text{volume under } e^{-r^2} \quad (r = \sqrt{x^2 + y^2})$$

We find this volume by the method of shells, which leads to the same integral as in the last problem. The shell or cylinder under e^{-r^2} at radius r has circumference $2\pi r$, thickness dr; (see Figure 9). Therefore $dV = e^{-r^2} 2\pi r dr$. In the range $0 \le r \le R$,

$$\int_0^R e^{-r^2} 2\pi r\,dr = -\pi e^{-r^2}\Big|_0^R = -\pi e^{-R^2} + \pi$$

When $R \to \infty, e^{-R^2} \to 0$,

$$V = \int_0^{\infty} e^{-r^2} 2\pi r\,dr = \pi \qquad \text{(same as in the darts problem)}$$

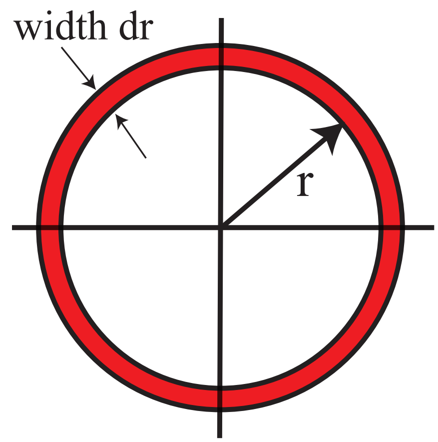

Figure 9: Area of annulus or ring, $(2\pi r)dr$.

Next, we will find V by a second method, the method of slices. Slice the solid along a plane where y is fixed. (See Figure 10). Call $A(y)$ the cross-sectional area. Since the thickness is dy (see Figure 11),

$$V = \int_{-\infty}^{\infty} A(y)\,dy$$

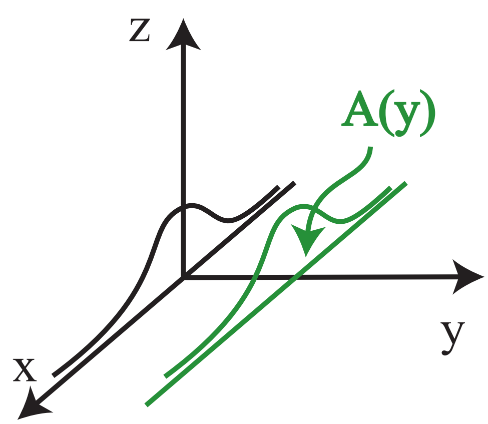

Figure 10: Slice $A(y)$.

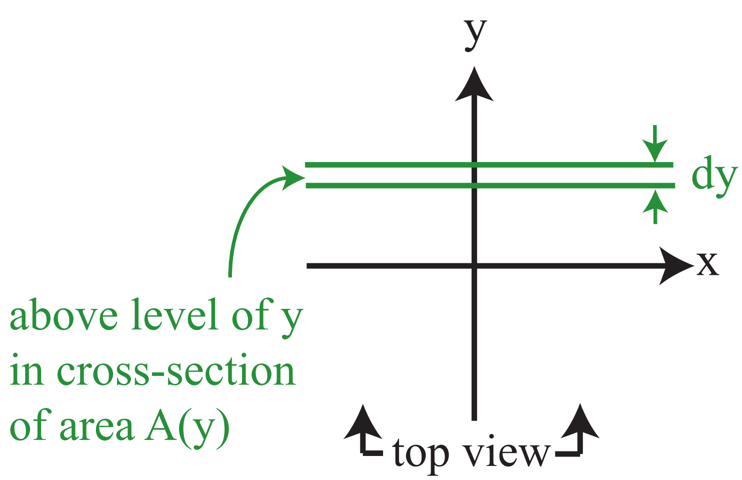

Figure 11: Top view of $A(y)$ slice.

To compute $A(y)$, note that it is an integral (with respect to dx)

$$A(y) = \int_{-\infty}^{\infty} e^{-r^2}\,dx = \int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx = e^{-y^2}\int_{-\infty}^{\infty} e^{-x^2}\,dx = e^{-y^2}Q$$

Here, we have used $r^2 = x^2 + y^2$ and

$$e^{-x^2-y^2} = e^{-x^2}e^{-y^2}$$

and the fact that y is a constant in the $A(y)$ slice (see Figure 12). In other words,

$$\int_{-\infty}^{\infty} ce^{-x^2}\,dx = c\int_{-\infty}^{\infty} e^{-x^2}\,dx \quad \text{with} \quad c = e^{-y^2}$$

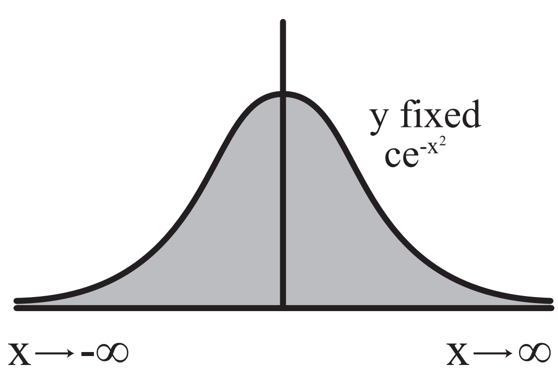

Figure 12: Side view of $A(y)$ slice.

It follows that

$$V = \int_{-\infty}^{\infty} A(y)\,dy = \int_{-\infty}^{\infty} e^{-y^2} Q\,dy = Q \int_{-\infty}^{\infty} e^{-y^2}\,dy = Q^2$$

Indeed,

$$Q = \int_{-\infty}^{\infty} e^{-x^2}\,dx = \int_{-\infty}^{\infty} e^{-y^2}\,dy$$

because the name of the variable does not matter. To conclude the calculation read the equation backwards:

$$\pi = V = Q^2 \implies \boxed{Q = \sqrt{\pi}}$$

We can rewrite $Q = \sqrt{\pi}$ as

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2}\,dx = 1$$

An equivalent rescaled version of this formula (replacing x with $x/\sqrt{2}\sigma$)is used:

$$\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-x^2/2\sigma^2}\,dx = 1$$

This formula is central to probability and statistics. The probability distribution $\frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2}$ on $-\infty < x < \infty$ is known as the normal distribution, and $\sigma > 0$ is its standard deviation.