18.01 Single Variable Calculus
Fall 2006

18.01 UNIT 2 REVIEW; Fall 2007

The central theme of Unit 2 is that knowledge of f' (and sometimes f'') tells us something about f itself. This is even true of our first topic, approximation. For instance, knowing that $f(x) = e^x$ satisfies $f(0) = 1$ and $f'(0) = 1$, we can say

$$e^x \approx 1 + x \quad \text{provided} \quad x \approx 0$$

The linear function $1 + x$ is much simpler than e^x, so $f(0)$ and $f'(0)$ give us a (very) simplified picture of our function, useful only near near 0. For more detail, use the quadratic approximation,

$$e^x \approx 1 + x + x^2/2 \quad \text{provided} \quad x \approx 0$$

(still only works well near 0)

The second and third practice exams are actual tests from previous years. The exam this year is similar to the one from 2006 posted at our site. It has 6 questions covering the following topics. (No Newton's method, but there is a seventh, extra credit problem.)

1. Linear and/or quadratic approximations
2. Sketch a graph $y = f(x)$
3. Max/min
4. Related rates
5. Find antiderivatives and solve a differential equation by separating variables
6. Mean value theorem.

Remarks.

1. Recall that linear [and quadratic] approximation is

$$f(x) \approx f(a) + f'(a)(x - a) \ [+(f''(a)/2)(x - a)^2]$$

2. You should expect to graph a function $y = f(x)$, where $f(x)$ is a rational function (ratio of polynomials).

Warnings:

a) When asked to label the critical point on the graph, find and mark the point (a, b). In lecture we called $x = a$ the critical point and $y = b$ the critical value, and this is what is used in 18.02, and elsewhere. But for this exam (and this is just an inconsistency in language that you will have to tolerate) the words "critical point" refer to the point on the graph (a, b), not the number a and the point on the x-axis. The same applies to inflection points.

b) $y = 1/(x - 1)$ is decreasing on the intervals $-\infty < x < 1$ and $1 < x < \infty$, but it is **not decreasing** on the interval $-\infty < x < \infty$. Draw the graph to see.

You cannot just use the fact that $y' = -1/(x - 1)^2 < 0$ because there is a point in the middle at which y is not differentiable — and not even continuous. So the mean value theorem does not apply.

c) Similarly, $y = 1/(x - 1)^2$ is concave up on $-\infty < x < 1$ and $1 < x < \infty$, but it is **not concave up** on the interval $-\infty < x < \infty$. Here $y'' = 6/(x - 1)^4 > 0$, but there is a singularity in the middle. Plot the graph yourself to see.

3. The mean value theorem says that if f is differentiable, then for some c, $a < c < x$,

$$f(x) = f(a) + f'(c)(x - a)$$

It is used as follows. Suppose that $m < f'(c) < M$ on the interval $a < c < x$, then

$$f(x) = f(a) + f'(c)(x - a) < f(a) + M(x - a)$$

Similarly,

$$f(x) = f(a) + f'(c)(x - a) > f(a) + m(x - a)$$

Put another way, if $\Delta f = f(x) - f(a)$ and $\Delta x = x - a$, and $m < f'(c) < M$ for $a < c < x$, then

$$m\Delta x < \Delta f < M\Delta x$$

More consequences of the mean value theorem.

A function f is called increasing (also called strictly increasing) if $x > a$ implies $f(x) > f(a)$. The reasoning above with $m = 0$ shows that if $f' > 0$, then f is increasing. Similarly if $f' < 0$, then f is decreasing. We use these facts every time we sketch a graph of a function or find a maximum or minimum.

A similar discussion works when the inequality is not strict. If $m \le f'(c) \le M$ for $a < c < x$, then

$$f(a) + m(x - a) \le f(x) \le f(a) + M(x - a)$$

A function is called nondecreasing if $x > a$ implies $f(x) \ge f(a)$. If $f' \ge 0$, then the inequality above shows that f is nondecreasing. Conversely, if the function is nondecreasing and differentiable, then $f' \ge 0$. Similarly, differentiable functions are nonincreasing if and only if they satisfy $f' \le 0$.

Key corollary to the mean value theorem: $f' = g'$ implies $f - g$ is constant.

In Unit 2, we have found that information about f' gives information about f. In particular, knowing a starting value for a function and its rate of change determines the function. A seemingly obvious example is that if $f' = 0$ for all x, then f is constant. If this were not true, then the mathematical notion of derivative would fail to coincide with our intuitive notion of what rate of change and cause and effect mean.

But this fundamental fact needs a proof. Derivatives are instantaneous quantities, obtained as limits. It is the mean value theorem that allows us to pass in rigorous mathematical fashion from the infinitesimal to the practical, human scale. Here is the proof. If $f' = 0$, then one can take $m = M = 0$ in the inequalities above, and conclude that $f(x) = f(a)$. In other words, f is constant. As an immediate consequence, if $f' = g'$, then f and g differ by a constant. (Apply the previous argument to the function $f - g$, whose derivative is 0.) This basic fact will lead us shortly to what is known as the fundamental theorem of calculus.