Lecture 9. September 29, 2005

Homework. Problem Set 2 all of Part I and Part II.

Practice Problems. Course Reader: 2B-1, 2B-2, 2B-4, 2B-5.

1. Application of the Mean Value Theorem. A real-world application of the Mean Value Theorem is *error analysis*. A device accepts an input signal x and returns an output signal y. If the input signal is always in the range $-1/2 \leq x \leq 1/2$ and if the output signal is,

$$y = f(x) = \frac{1}{1 + x + x^2 + x^3},$$

what precision of the input signal x is required to get a precision of $\pm 10^{-3}$ for the output signal?

If the ideal input signal is $x = a$, and if the *precision* is $\pm h$, then the actual input signal is in the range $a - h \leq x \leq a + h$. The precision of the output signal is $|f(x) - f(a)|$. By the Mean Value Theorem,

$$\frac{f(x) - f(a)}{x - a} = f'(c),$$

for some c between a and x. The derivative $f'(x)$ is,

$$f'(x) = \frac{-(3x^2 + 2x + 1)}{(1 + x + x^2 + x^3)^2}.$$

For $-1/2 \leq x \leq 1/2$, this is bounded by,

$$|f'(x)| \leq \frac{3(1/2)^2 + 2(1/2) + 1}{[1 + (-1/2) + (-1/2)^2 + (-1/2)^3]^2} = 7.04.$$

Thus the Mean Value Theorem gives,

$$|f(x) - f(a)| = |f'(c)||x - a| \leq 7.04|x - a| \leq 7.04h.$$

Therefore a precision for the input signal of,

$$h = \boxed{10^{-3}/7.04} \approx 10^{-4}$$

guarantees a precision of 10^{-3} for the output signal.

2. First derivative test. A function $f(x)$ is *increasing*, respectively *decreasing*, if $f(a)$ is less than $f(b)$, resp. greater than $f(b)$, whenever a is less than b. In symbols, f is increasing, respectively decreasing, if

$$f(a) < f(b) \text{ whenever } a < b, \text{ resp. } f(a) > f(b) \text{ whenever } a < b.$$

If $f(a)$ is less than or equal to $f(b)$, resp. greater than or equal to $f(b)$, whenever a is less than b, then $f(x)$ is *non-decreasing*, resp. *non-increasing*. If $f(x)$ is increasing, the graph rises to the right. If $f(x)$ is decreasing, the graph rises to the left.

If $f'(a)$ is positive, the *First Derivative Test* guarantees that $f(x)$ is increasing for all x sufficiently close to a. If $f'(a)$ is negative, the First Derivative Test guarantees that $f(x)$ is decreasing for all x sufficiently close to a.

Example. For the function $y = x^3 + x^2 - x - 1$, determine where y is increasing and where y is decreasing.

The derivative is,

$$y' = 3x^2 + 2x - 1 = (3x - 1)(x + 1).$$

Thus the derivative of y changes sign only at the points $x = -1$ and $x = 1/3$. By testing random elements, y' is positive for $x > 1/3$, it is negative for $-1 < x < 1/3$, and it is positive for $x < -1$. Therefore, by the First Derivative Test, y is increasing for $x < -1$, y is decreasing for $-1 < x < 1/3$, and y is increasing for $x > 1/3$.

3. Extremal points. If $f(x) \leq f(a)$ for all x near a, then x is a *local maximum*. If $f(x) \geq f(a)$ for all x near a, then x is a *local minimum*. Because of the First Derivative Test, if $f'(a) > 0$ and f is defined to the right of a, the graph of f rises to the right of a. Thus a is not a local maximum. Similarly, if $f'(a) < 0$ and f is defined to the left of a, the graph of f rises to the left of a. Thus a is not a local maximum. In particular, if f is defined to both the right and left of a, if $f'(a)$ is defined, and if a is a local maximum, then $f'(a)$ equals 0. Similarly, if f is defined to both the right and left of a, if $f'(a)$ is defined, and if a is a local minimum, then $f'(a)$ equals 0.

A point a where $f'(a)$ is defined and equals 0 is a *critical point*. By the last paragraph, if $x = a$ is a local maximum of f, respectively a local minimum of f, then one of the following holds.

(i) The function $f(x)$ is discontinuous at a.

(ii) The function $f(x)$ is continuous at a, but $f'(a)$ is not defined.

(iii) The point a is a left endpoint of the interval where f is defined, and $f'(a) \leq 0$, resp. $f'(a) \geq 0$.

(iv) The point a is a right endpoint of the interval where f is defined, and $f'(a) \geq 0$, rexp. $f'(a) \leq 0$.

(v) The function f is defined to the left and right of a, and $f'(a)$ equals 0. In other words, a is a critical point of f.

Example. For the function $y = x^3 + x^2 - x - 1$, the critical points are $x = -1$ and $x = 1/3$. By examining where y is increasing and decreasing, $x = -1$ is a local maximum and $x = 1/3$ is a local minimum.

The plurals of "maximum" and "minimum" are "*maxima*" and "*minima*". Together, local maxima and local minima are called *extremal points*, or *extrema*. These are points where f takes on an extreme value, either positive or negative. A point where f achieves its maximum value among *all* points where f is defined is a *global maximum* or *absolute maximum*. A point where f achieves its minimum value among *all* points where f is defined is a *global minimum* or *absolute minimum*.

4. Concavity and the Second Derivative Test. For a differentiable function f, every "interior" extremal point is a critical point of f. But not every critical point of f is an extremal point.

Example. The function $f(x) = x^3$ has a critical point at $x = 0$. But $f(x)$ is everywhere increasing, thus $x = 0$ is not an extremal point of f.

When is a critical point an extremal point? When is it a local maximum? When is it a local minimum? This is closely related to the *concavity* of f. A function $f(x)$ is *concave up*, respectively *concave down*, if no secant line segment to $f(x)$ crosses below the graph of f, resp. above the graph of f. In symbols, f is concave up, resp. concave down, if

$$(f(c) - f(a))/(c - a) \leq (f(b) - f(a))/(b - a) \text{ whenever } a < c < b,$$

$$\text{resp. } (f(c) - f(a))/(c - a) \geq (f(b) - f(a))/(b - a) \text{ whenever } a < c < b.$$

For a differentiable function f, this equation is close to,

$$f'(c) \leq f'(b) \text{ whenever } a < c < b,$$

$$\text{resp. } f'(c) \geq f'(b) \text{ whenever } a > c > b.$$

This precisely says that f' is non-decreasing, resp. f' is non-increasing. If f' is non-decreasing, resp. non-increasing, then f is concave up, resp. concave down. Applying the First Derivative Test to determine when f' is increasing, resp. decreasing, gives the *Second Derivative Test*: If $f''(a) > 0$, then f is concave up near $x = a$; if $f''(a) < 0$ then f is concave down near $x = a$.

If f is concave up near a critical point, the critical point is a local minimum. If f is concave down near a critical point, the critical point is a local maximum. Combined with the Second Derivative Test, this gives a test for when a critical point is a local maximum or local minimum: If $f'(a)$ equals 0 and $f''(a) < 0$, then $x = a$ is a local maximum. If $f'(a)$ equals 0 and $f''(a) > 0$, then $x = a$ is a local minimum.

Example. For $y = x^3 + x^2 - x - 1$, the second derivative is $y'' = 6x + 2$. Since $y''(-1) = -4$ is negative, the critical point $x = -1$ is a local maximum. Since $y''(1/3) = 4$ is positive, $x = 1/3$ is a local minimum.

5. Inflection points. If f is differentiable, but for every neighborhood of a, f is neither concave up nor concave down on the entire neighborhood, then a is an *inflection point*. If $f''(a)$ is defined, the Second Derivative Test says that $f''(a)$ must equal 0. Except in pathological cases, an inflection point is a point where f is concave up to one side of f, and concave down to the other side of f.

Example. For $y = x^3 + x^2 - x - 1$, the second derivative $y'' = 6x + 2$ is negative for $x < -1/3$ and is positive for $x > 1/3$. By the Second Derivative Test, y is concave down for $x < -1/3$ and y is concave up for $x > -1/3$. Therefore $x = -1/3$ is an inflection point for y.