Lecture 4. September 15, 2005

Homework. No new problems.

Practice Problems. Course Reader: 1F-1, 1F-6, 1F-7, 1F-8.

1. Product rule example. For $u = \sqrt{3x+1}$, what is $u'(x)$? Since $u \cdot u = 3x + 1$, $(u \cdot u)' = (3x+1)' = 3$. By the product rule, $(u \cdot u)' = u' \cdot u + u \cdot u' = 2uu'$. Thus solving,

$$u'(x) = 3/(2u) = \boxed{3(3x+1)^{-1/2}/2}.$$

2. The derivative of u^n. From above, $(u^2)'$ equals $2uu'$. By a similar computation, $(u^3)'$ equals $3u^2u'$. This suggests a pattern,

$$\frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx}.$$

This can be proved by induction on n. For $n = 1, 2$ and 3, it was checked. Let n be a particular integer (for instance, 70119209472933054321). For that integer, suppose the result is known,

$$\frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx}.$$

The goal is to prove the result for $n + 1$, that is,

$$\frac{d(u^{n+1})}{dx} = (n + 1)u^n\frac{du}{dx}.$$

Let $v = u^n$. Then u^{n+1} equals uv. So, by the product rule,

$$\frac{d(u^{n+1})}{dx} = \frac{d(uv)}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}.$$

Plugging in $v = u^n$, this is,

$$\frac{d(u^{n+1})}{dx} = \frac{du}{dx} \cdot (u^n) + u\frac{d(u^n)}{dx}.$$

By the induction hypothesis, $d(u^n)/dx$ equals $nu^{n-1}(du/dx)$. Plugging in,

$$\frac{d(u^{n+1})}{dx} = \frac{du}{dx} \cdot (u^n) + u(nu^{n-1}\frac{du}{dx}).$$

This simplfies to,

$$\frac{d(u^{n+1})}{dx} = u^n\frac{du}{dx} + nu^n\frac{du}{dx} = (n + 1)u^n\frac{du}{dx}.$$

Thus, the result for $n + 1$ follows from the result for n. By induction, the result holds for every n.

3. The derivative of x^a, a a fraction. Let a be a fraction m/n and let $u(x)$ be x^a. Then u^n equals x^m. Thus,

$$\frac{d(u^n)}{dx} = \frac{d(x^m)}{dx},$$

which equals mx^{m-1}. By the above, $d(u^n)/dx$ equals $nu^{n-1}(du/dx)$. Thus,

$$nu^{n-1}\frac{du}{dx} = mx^{m-1}.$$

Solving for du/dx,

$$\frac{du}{dx} = \frac{mx^{m-1}}{nu^{n-1}} = \frac{mx^{m-1}}{n(x^{m/n})^{n-1}}.$$

One of the basic rules of exponents is that $(a^b)^c$ equals a^{bc}. Thus the denominator $n(x^{m/n})^{n-1}$ equals $nx^{m/n(n-1)}$, which equals $nx^{m-m/n}$. Thus,

$$\frac{du}{dx} = \frac{mx^{m-1}}{nx^{m-m/n}} = \frac{m}{n}x^{m-1} \cdot x^{m/n-m}.$$

Another basic rule of exponents is that $a^b \cdot a^c$ equals a^{b+c}. Thus,

$$\frac{du}{dx} = \frac{m}{n} x^{(m-1)+(m/n-m)} = \frac{m}{n} x^{m/n-1}.$$

Remembering that m/n is just a, and $u(x)$ is x^a, this finally gives,

$$\frac{d(x^a)}{dx} = ax^{a-1}.$$

4. The chain rule. Let y be a function of x, $y = f(x)$, and let u be a function of y, $u = g(y)$. Then u is a function of x, $u = g(f(x))$. This function is a **composite function**, and is denoted by,

$$(g \circ f)(x) = g(f(x)).$$

What is the derivative of a composite function? The claim is that,

$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x).$$

This is often easier to remember in the form,

$$\frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx}.$$

This also suggests the proof,

$$(g \circ f)'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta y} \cdot \frac{\Delta y}{\Delta x},$$

where y_0 equals $f(x_0)$, u_0 equals $g(y_0) = g(f(x_0))$, Δy equals $f(x_0+\Delta x) - f(x_0) = f(x_0+\Delta x) - y_0$, and Δu equals $g(y_0+\Delta y) - g(y_0) = g(f(x_0+\Delta x)) - g(f(x_0))$. So long as Δy is nonzero, the fraction in the limit is defined. And, as Δx approaches 0, also Δy approaches 0. Thus the limit breaks up as,

$$(g \circ f)'(x_0) = \lim_{\Delta y \to 0} \frac{\Delta u}{\Delta y} \cdot \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = g'(y_0) \cdot f'(x_0).$$

Thus $(g \circ f)'(x_0)$ equals $g'(f(x_0))f'(x_0)$.

Example. Let $y(x)$ equals $1 + x^2$, and let $u(y)$ equal $1/y = y^{-1}$. Then $y'(x) = 0 + 2x = 2x$ and $u'(y) = -y^{-2}$. Thus, by the chain rule,

$$\frac{d}{dx}\left(\frac{1}{1+x^2}\right) = \frac{-1}{y^2}(2x) = \frac{-2x}{(1+x^2)^2}.$$

5. Implicit differentiation. This method has already been used many times. Given a function $y(x)$ satisfying some equation involving both x and y, formally differentiate each side of the equation with respect to x and then try to solve for y'.