

Lecture 27. November 22, 2005

Homework. Problem Set 7 Part II: Problem 2.

Practice Problems. Course Reader: 5F-2, 5F-3, 5F-4, 5F-5.

1. Integration by parts. The differential form of the product rule is,

$$d(uv) = u dv + v du.$$

An equivalent form is,

$$u dv = d(uv) - v du.$$

This gives a very useful antidifferentiation formula,

$$\int u dv = uv - \int v du.$$

This formula is *integration by parts*.

Example. Compute the antiderivative of,

$$\int x \cos(x) dx.$$

Set u to be x and dv to be $\cos(x)dx$. Then u, v, du and dv are,

$$\begin{aligned} u &= x & dv &= \cos(x)dx, \\ du &= dx & v &= \sin(x) \end{aligned}$$

Using integration by parts,

$$\begin{aligned} \int u dv &= uv - \int v du, \\ \int x \cos(x) dx &= x \sin(x) - \int \sin(x) dx. \end{aligned}$$

The new integral is easy to evaluate. Altogether this gives,

$$\int x \cos(x) dx = x \sin(x) + \cos(x) + C.$$

Because it is much easier to differentiate than the antiderivative, it is a good idea to check your answer.

2. How to use integration by parts. The goal of integration by parts is to replace a complicated integral, $\int u dv$, by a simpler integral $\int v du$. What this usually means is that du should be simpler than u , and v should be no more complicated than dv . This was the case in the last example. However, occasionally this is not the case.

Example. Use integration by parts to compute the antiderivative,

$$\int \ln(x) dx.$$

There is very little choice here, if we are to use only integration by parts. Set u to be $\ln(x)$ and set dv to be dx . Then u , v , du and dv are,

$$\begin{aligned} u &= \ln(x), & dv &= dx \\ du &= dx/x, & v &= x \end{aligned}$$

Using integration by parts,

$$\begin{aligned} \int u dv &= uv - \int v du, \\ \int \ln(x) dx &= x \ln(x) - \int dx. \end{aligned}$$

The new integral is easy to evaluate. Altogether this gives,

$$\int \ln(x) dx = x \ln(x) - x + C.$$

Notice this example does not follow the general rule. The integral $v = x$ is strictly more complicated than $dv = dx$. However, $du = dx/x$ is much simpler than $u = \ln(x)$. So $v du = dx$ is simpler than $u dv = \ln(x) dx$. The lesson is to be flexible when antiderivating. Try different things, and see which one works. For example, another approach to this problem, which ultimately comes down to integration by parts again, is to make an inverse substitution,

$$x = e^t, \quad dx = e^t dt.$$

The new integral is,

$$\int \ln(x) dx = \int t e^t dt.$$

Set $u = t$ and $dv = e^t dt$. Then u , v , du and dv are,

$$\begin{aligned} u &= t, & dv &= e^t dt \\ du &= dt, & v &= e^t \end{aligned}$$

Using integration by parts,

$$\int u dv = uv - \int v du,$$

$$\int te^t dt = te^t - \int e^t dt.$$

The new integral is easy to evaluate. Altogether this gives,

$$\int te^t dt = te^t - e^t + C.$$

Back-substituting for x gives,

$$\int \ln(x) dx = x \ln(x) - x + C.$$

This agrees with the earlier answer.

2. Reduction formulas. It often happens that an integral can be computed only by repeated application of integration by parts. It sometimes happens that integration by parts gives the induction step to solve infinitely many integrals. In this case, the formula given by integration by parts is called a *reduction formula*.

Example. Use integration by parts to give a reduction formula for,

$$\int [\ln(x)]^n dx.$$

Now there is much more choice for u and dv . The simplest choice is to set $u = [\ln(x)]^n$ and $dv = dx$. Then u , v , du and dv are,

$$u = [\ln(x)]^n, \quad dv = dx$$

$$du = n[\ln(x)]^{n-1}/x dx, \quad v = x$$

Using integration by parts,

$$\int u dv = uv - \int v du,$$

$$\int [\ln(x)]^n dx = x[\ln(x)]^n - n \int [\ln(x)]^{n-1} dx.$$

The new integral is simpler than the original integral. And repeated application of the formula eventually leads to a formula for the integral. Thus this is a reduction formula. For instance, this gives,

$$\int [\ln(x)]^2 dx = x[\ln(x)]^2 - 2 \int \ln(x) dx.$$

The new integral was already computed. Altogether this gives,

$$\int [\ln(x)]^2 dx = x[\ln(x)]^2 - 2x \ln(x) + 2x + C.$$

Example. Use integration by parts to find a reduction formula for,

$$\int t^n e^t dt.$$

The simplest choice is to set $u = t^n$ and $dv = e^t dt$. Then u , v , du and dv are,

$$\begin{aligned} u &= t^n, & dv &= e^t dt \\ du &= nt^{n-1} dt, & v &= e^t \end{aligned}$$

Using integration by parts,

$$\begin{aligned} \int u dv &= uv - \int v du, \\ \int t^n e^t dt &= t^n e^t - n \int t^{n-1} e^t dt. \end{aligned}$$

Notice how similar this answer was to the answer of the previous example. The connection comes from the inverse substitution,

$$x = e^t, \quad dx = e^t dt,$$

so that,

$$\int [\ln(x)]^n dx = \int t^n e^t dt.$$

3. Advanced reduction formulas. Sometimes a reduction formula can only be obtained by repeatedly applying integration by parts or by using some other identity.

Example. Using integration by parts to find a reduction formula for,

$$\int [\sin(x)]^n dx, \quad n \geq 1.$$

One choice is to set $u = [\sin(x)]^{n-1}$ and to set $dv = \sin(x) dx$. Then u , v , du and dv are,

$$\begin{aligned} u &= [\sin(x)]^{n-1}, & dv &= \sin(x) dx \\ du &= (n-1)[\sin(x)]^{n-2} \cos(x) dx, & v &= -\cos(x). \end{aligned}$$

Using integration by parts,

$$\begin{aligned} \int u dv &= uv - \int v du, \\ \int [\sin(x)]^n dx &= -[\sin(x)]^{n-1} \cos(x) + (n-1) \int [\sin(x)]^{n-2} \cos^2(x) dx. \end{aligned}$$

At first blush, this is more complicated than the original integral since it involves both $\sin(x)$ and $\cos(x)$. But $\cos^2(x)$ equals $1 - \sin^2(x)$. This substitution gives,

$$\int [\sin(x)]^n dx = -[\sin(x)]^{n-1} \cos(x) + (n-1) \int [\sin(x)]^{n-2} dx - (n-1) \int [\sin(x)]^n dx.$$

This certainly seems circular: the new formula for the integral involves the integral we were looking for. However, bringing like terms to one side of the equation gives,

$$\int [\sin(x)]^n dx + (n-1) \int [\sin(x)]^n = -[\sin(x)]^{n-1} \cos(x) + (n-1) \int [\sin(x)]^{n-2} dx.$$

Cleaning this up a bit gives the reduction formula,

$$\int [\sin(x)]^n dx = -[\sin(x)]^{n-1} \cos(x)/n + (n-1)/n \int [\sin(x)]^{n-2} dx.$$

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