Lecture 24. November 15, 2005

Practice Problems. Course Reader: 5A-1, 5A-2, 5A-3, 5A-5, 5A-6.

1. Inverse functions. Let a, b, s and t be constants. Let $y = f(x)$ be a function defined on the interval,

$$a \leq x \leq b,$$

and whose values are in the interval,

$$s \leq y \leq t.$$

Does there exist a function $x = g(y)$ defined on the interval,

$$s \leq y \leq t,$$

whose values are in the interval,

$$a \leq x \leq b,$$

satisfying the two conditions,

$$g(f(x)) = x, \quad f(g(y)) = y \ ?$$

If such a function g exists, it is called an *inverse function* of f, and it is denoted by $f^{-1}(y)$. Also, the original function $f(x)$ is called *invertible*. There is some chance of confusion with the other use of "invertible", namely that $1/f(x)$ is always defined. We will be careful to specify the meaning of "invertible".

There are 2 necessary conditions for f to have an inverse function. Assume f has an inverse function g. Let x_1, x_2 be a pair of numbers in $[a, b]$. If $f(x_1)$ equals $f(x_2)$, then also,

$$x_1 = g(f(x_1)) = g(f(x_2)) = x_2,$$

i.e., x_1 equals x_2. In other words, two distinct inputs x_1 and x_2 give two distinct outputs $f(x_1)$ and $f(x_2)$. A function satisfying this condition is called *one-to-one*, because to every output, there is at most one input. This is the first necessary condition: every invertible function is one-to-one.

Next, for every number y in $[s, t]$, there is a number x in $[a, b]$ such that $y = f(x)$. In fact, just take x to be $g(y)$; then $f(x)$ equals $f(g(y))$, which equals y. A function satisfying this condition is called *onto*. This is the second necessary condition: every invertible function is onto.

Together, this says that an invertible function is one-to-one and onto. In fact, the converse is also true: every one-to-one and onto function is invertible. This is easy to check, but we will not prove it in this class.

Remark: In checking that f is one-to-one and onto, the choice of intervals $[a, b]$ and $[c, d]$ are vital. A simple example comes from $f(x) = \sin(x)$. For the interval $[-\pi/2, \pi, 2]$ and $[-1, 1]$, the function $f(x)$ is one-to-one and onto. But for many other choices of these intervals, the function is neither one-to-one nor onto.

2. The graph of an inverse function. How should we think of an inverse function? One way is graphically. The graph of the function $y = f^{-1}(x)$ is the same as the graph of $f(y) = x$. This is simply the usual graph of $y = f(x)$ with the roles of x and y reversed. What this translates to is, the graph of f^{-1} is the same as the graph of f with the roles of the x-axis and y-axis reversed. The simplest way to get the graph of $f^{-1}(x)$ is simply to reflect the graph of $f(x)$ through the 45° line $y = x$.

3. The inverse trigonometric functions. The function $\sin(x)$ is one-to-one and onto on $[-\pi/2, \pi/2]$, taking values in $[-1, 1]$. Thus there is an inverse function $\sin^{-1}(x)$ defined on the interval $[-1, 1]$, taking values in $[-\pi/2, \pi/2]$. The graph of $\sin^{-1}(x)$ is an increasing function whose lower left endpoint is $(-1, -\pi/2)$ and whose upper right endpoint is $(1, \pi/2)$.

The function $\cos(x)$ is one-to-one and onto on $[0, \pi]$, taking values in $[-1, 1]$. Thus there is an inverse function $\cos^{-1}(x)$ defined on the interval $[-1, 1]$, taking values in $[0, \pi]$. The graph of $\cos^{-1}(x)$ is a decreasing function whose upper left endpoint is $(-1, \pi)$ and whose lower right endpoing is $(1, 0)$.

The function $\tan(x)$ is one-to-one and onto on $(-\pi/2, \pi/2)$, taking values in the whole real line. Thus there is an inverse function $\tan^{-1}(x)$ defined on the whole real line, taking values in $(-\pi/2, \pi/2)$. The graph is an increasing function that is asymptotic to the line $y = -\pi/2$ as $x \to -\infty$, and asymptotic to the line $y = +\pi/2$ as $x \to +\infty$.

4. Derivatives of inverse functions. A particular simple formulation of the chain rule is the differential formulation,

$$df(u) = f'(u)du.$$

If f has an inverse function $g(x)$, let u be $g(x)$. Then this gives,

$$df(g(x)) = f'(g(x))dg(x).$$

On the other hand, $f(g(x))$ equals x. This gives the formula,

$$dx = f'(g(x))dg(x).$$

Solving for dg/dx gives,

$$\frac{d}{dx}(g(x)) = \boxed{1/f'(g(x)).}$$

This is the formula for the derivative of an inverse function.

In fact, we have seen this formula before. It is how we computed the derivative of $\ln(x)$, the inverse function of e^x:

$$\frac{d}{dx}(\ln(x)) = \frac{1}{e^{\ln(x)}} = \frac{1}{x}.$$

5. Derivatives of the inverse trigonometric functions. Because the derivative of $\sin(x)$ is $\cos(x)$, the formula above gives,

$$\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\cos(\sin^{-1}(x))}.$$

This isn't very useful. A simple argument makes it much more useful. Denote $\sin^{-1}(x)$ by θ. Thus $\sin(\theta) = x$. Also, the formula for the derivative is a bit simpler,

$$\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\cos(\theta)}.$$

By the Pythagorean theorem,

$$\sin^2(\theta) + \cos^2(\theta) = 1.$$

Solving gives,

$$\cos(\theta) = \sqrt{1 - \sin^2(\theta)} = \sqrt{1 - x^2}.$$

This gives a very useful formula for the derivative of $\sin^{-1}(x)$,

$$\frac{d}{dx}(\sin^{-1}(x)) = \boxed{1/\sqrt{1 - x^2}.}$$

There is a very similar derivation that,

$$\frac{d}{dx}(\cos^{-1}(x)) = \boxed{-1/\sqrt{1 - x^2}.}$$

This looks remarkably similar to the previous formula. In particular, this gives,

$$\frac{d}{dx}(\sin^{-1}(x) + \cos^{-1}(x)) = \frac{1}{\sqrt{1-x^2}} + \frac{-1}{\sqrt{1-x^2}} = 0.$$

Therefore the sum is a constant function. Checking at $x = 0$ gives the value of this constant function,

$$\boxed{\sin^{-1}(x) + \cos^{-1}(x) = \pi/2.}$$

Finally, because the derivative of $\tan(x)$ is $\sec^2(x)$, the formula gives,

$$\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{\sec^2(\tan^{-1}(x))}.$$

Again introduce $\theta = \tan^{-1}(x)$. Then the formula for the derivative is,

$$\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{\sec^2(\theta)}.$$

But the Pythagorean theorem implies,

$$\sec^2(\theta) = 1 + \tan^2(\theta) = 1 + x^2.$$

This finally gives a very useful formula for the derivative of $\tan(x)$,

$$\frac{d}{dx}(\tan^{-1}(x)) = \boxed{1/(1+x^2).}$$

Notice, in particular, that the denominator is never zero. This is closely related to the fact that $\tan^{-1}(x)$ is defined on the entire real line.

6. Hyperbolic trigonometric functions. The trigonometric functions are very useful for discussing point on the unit circle $x^2 + y^2 = 1$, because the circle is the parametric curve,

$$\begin{cases} x &= \cos(\theta), \\ y &= \sin(\theta) \end{cases}$$

Are there analogous continuous functions for the points on the hyperbola $x^2 - y^2 = 1$?

At first blush, the answer is no. The problem is that the hyperbola has two parts: one part is in the half-plane where $x > 0$, and the other part is in the half-plane where $x < 0$. Because of the intermediate value theorem, a continuous function $x = f(t)$ cannot jump from $x > 0$ to $x < 0$ or vice versa without crossing $x = 0$. Thus, refine the question: Are there continuous functions for the part of the hyperbola in the half-plane where $x > 0$?

The answer to this question is yes. The corresponding functions are called *hyperbolic trigonometric functions* or, more often, simply *hyperbolic functions*. They are defined as follows,

$$\cosh(t) = \frac{1}{2}(e^t + e^{-t}),$$

$$\sinh(t) = \frac{1}{2}(e^t - e^{-t}),$$

$$\tanh(t) = \frac{\sinh(t)}{\cosh(t)} = \frac{e^t - e^{-t}}{e^t + e^{-t}},$$

$$\text{sech}(t) = \frac{1}{\cosh(t)} = \frac{2}{e^t + e^{-t}},$$

$$\text{csch}(t) = \frac{1}{\sinh(t)} = \frac{2}{e^t - e^{-t}},$$

and,

$$\coth(t) = \frac{1}{\tanh(t)} = \frac{\cosh(t)}{\sinh(t)} = \frac{e^t + e^{-t}}{e^t - e^{-t}}.$$

The first observation is that,

$$\cosh^2(t) = \frac{1}{4}(e^t + e^{-t})^2 = \frac{1}{4}(e^{2t} + 2 + e^{-2t}),$$

$$\sinh^2(t) = \frac{1}{4}(e^t - e^{-t})^2 = \frac{1}{4}(e^{2t} - 2 + e^{-2t}).$$

Taking the difference of these, most of the terms cancel,

$$\cosh^2(t) - \sinh^2(t) = \frac{1}{4}((2) - (-2)) = \frac{4}{4} = 1.$$

This proves that the parametric curve,

$$\begin{cases} x = \cosh(t), \\ y = \sinh(t) \end{cases}$$

is contained in the right-half of the hyperbola $x^2 - y^2 = 1$. We will see next time that there is an inverse function of $\sinh(t)$, from which it follows that *every* point in the right-half of the hyperbola occurs for *exactly* one value of t. Thus the parametric curve exactly traces out the right-half of the hyperbola.

7. The derivatives of the hyperbolic functions. The derivatives of the hyperbolic functions are straightforward. The formulas are very similar to the formulas in the trigonometric case, but slightly different. Try not to confuse them.

$$\frac{d}{dx}(\sinh(x)) = \boxed{\cosh(x).}$$

$$\frac{d}{dx}(\cosh(x)) = \boxed{\sinh(x).}$$

$$\frac{d}{dx}(\tanh(x)) = \frac{1}{\cosh^2(x)}(\cosh(x)\cdot\cosh(x) - \sinh(x)\cdot\sinh(x)) = \frac{1}{\cosh^2(x)} = \boxed{\text{sech}^2(x).}$$