18.01 Calculus

Jason Starr

Fall 2005

Lecture 23. November 8, 2005

Homework. Problem Set 6 Part I: (i) and (j); Part II: Problem 2.

Practice Problems. Course Reader: 4I-1, 4I-4, 4I-6.

1. Tangent lines to parametric curves. This short section was not explicitly discussed for general parametric curves. It was discussed for polar curves, which are a special collection of parametric curves.

/ /

Given a parametric curve,

$$\begin{cases} x &= f(t), \\ y &= g(t), \end{cases}$$

what is the slope of the tangent line at $(f(a), g(a))$? The relevant differentials are,

$$dx = f'(t)dt, \quad dy = g'(t)dt.$$

If $g'(a)$ is nonzero, then the slope of the tangent line is,

$$\frac{dy}{dx}(a) = \left.\frac{f'(t)dt}{g'(t)dt}\right|_{t=a} = \frac{f'(a)}{g'(a)}.$$

In particular, for a function $r = r(\theta)$, the associated polar curve is,

$$\begin{cases} x &= r(\theta)\cos(\theta), \\ y &= r(\theta)\sin(\theta) \end{cases}$$

Thus the differentials are,

$$\begin{aligned} dx &= [r'(\theta)\cos(\theta) - r(\theta)\sin(\theta)]d\theta, \\ dy &= [r'(\theta)\sin(\theta) + r(\theta)\cos(\theta)]d\theta. \end{aligned}$$

Therefore the slope of the tangent line is,

$$\frac{dy}{dx} = \frac{r'(\theta)\sin(\theta) + r(\theta)\cos(\theta)}{r'(\theta)\cos(\theta) - r(\theta)\sin(\theta)}.$$

2. Tangent lines for polar curves. Although the formula above is perfectly correct, it is a bit long to remember. There is a slightly different packaging that is much easier to remember. Define α to be the angle from the horizontal ray emanating from $(x(\theta), y(\theta))$ in the positive x-direction, and the tangent line. To be precise, there are two such angles, differing by π. The defining equation for α is,

$$\tan(\alpha) = \frac{dy}{dx}.$$

And, of course,

$$\tan(\theta) = \frac{y}{x}.$$

Define ψ to be the difference between α and θ,

$$\psi = \alpha - \theta.$$

The angle addition/subtraction formulas for $\tan(\theta)$ are,

$$\tan(\phi_1 + \phi_2) = \frac{\tan(\phi_1) + \tan(\phi_2)}{1 - \tan(\phi_1)\tan(\phi_2)}, \quad \tan(\phi_1 - \phi_1) = \frac{\tan(\phi_1) - \tan(\phi_2)}{1 + \tan(\phi_1)\tan(\phi_2)}.$$

Therefore,

$$\tan(\psi) = \tan(\alpha - \theta) = \frac{\tan(\alpha) - \tan(\theta)}{1 + \tan(\alpha)\tan(\theta)}.$$

Substituting in the equations for $\tan(\theta)$ and $\tan(\alpha)$ from above gives,

$$\tan(\psi) = \frac{(dy/dx) - (y/x)}{1 + (y/x)(dy/dx)}.$$

To simplify this, imagine multiplying both numerator and denominator by xdx and manipulate formally,

$$\tan(\psi) = \frac{xdy - ydx}{xdx + ydy}.$$

The actual justification of this is a little more involved, but the formal manipulation leads to the correct equation.

To compute the denominator in the expression, differentiate both sides of,

$$r^2 = x^2 + y^2,$$

to get,

$$2rdr = 2xdx + 2ydy,$$

or equivalently,

$$xdx + ydy = r(\theta)r'(\theta)d\theta.$$

To compute the numerator in the expression, differentiate both sides of,

$$\tan(\theta) = \frac{y}{x},$$

to get,

$$\sec^2(\theta)d\theta = \frac{dy}{x} - \frac{ydx}{x^2} = \frac{1}{x^2}(xdy - ydx).$$

Now substitute $x = r\cos(\theta)$ in the denominator to get,

$$\sec^2(\theta)d\theta = \frac{1}{r^2\cos^2(\theta)}(xdy - ydx) = \frac{\sec^2(\theta)}{r^2}(xdy - ydx).$$

Cancelling $\sec^2(\theta)$ and multiplying both sides by r^2 gives,

$$xdy - ydx = r^2 d\theta.$$

Thus the fraction for $\tan(\psi)$ is,

$$\tan(\psi) = \frac{xdy - ydx}{xdx + ydy} = \frac{r^2 d\theta}{rr' d\theta}.$$

Simplifying gives,

$$\tan(\psi) = \boxed{r(\theta)/r'(\theta)}.$$

Example. Consider the cardioid, discussed in recitation,

$$r(\theta) = a(1 + \cos(\theta)).$$

The formula for ψ is,

$$\tan(\psi) = \frac{r}{r'} = \frac{a(1 + \cos(\theta))}{-a\sin(\theta)} = \frac{1 + \cos(\theta)}{-\sin(\theta)}.$$

To simplify this, write $\theta = 2(\theta/2)$ and use the double-angle formulas to get,

$$\frac{1 + \cos(2(\theta/2))}{-\sin(2(\theta/2))} = \frac{1 + (\cos^2(\theta/2) - \sin^2(\theta/2))}{-2\sin(\theta/2)\cos(\theta/2)}.$$

Replacing $1 - \sin^2(\theta/2)$ in the numerator by $\cos^2(\theta/2)$, this simplfies to,

$$\frac{2\cos^2(\theta/2)}{-2\sin(\theta/2)\cos(\theta/2)} = -\cot(\theta/2).$$

Of course there is an identity,

$$-\cot(u) = \tan(u - \pi/2).$$

Altogether, this gives,

$$\tan(\psi) = -\cot(\theta/2) = \tan(\theta/2 - \pi/2).$$

Therefore,

$$\psi = \boxed{(\theta - \pi)/2.}$$

Since α equals $\theta + \psi$, this gives,

$$\alpha = \boxed{(3\theta - \pi)/2.}$$

In particular, the angle of the tangent line to the cardioid at $\theta = \pi/2$ is $\alpha = \pi/4$.

3. Arc length in polar coordinates. As discussed previously, the formula for arc length of a parametric curve is,

$$ds = \sqrt{(dx/dt)^2 + (dy/dt)^2}dt.$$

In the case of a parametric curve, this becomes a bit simpler. The differentials are,

$$dx = (r'(\theta)\cos(\theta) - r(\theta)\sin(\theta))d\theta,$$
$$dy = (r'(\theta)\sin(\theta) + r(\theta)\cos(\theta))d\theta.$$

Squaring gives,

$$(dx)^2 = ((r')^2\cos^2(\theta) - 2rr'\sin(\theta)\cos(\theta) + r^2\sin^2(\theta))(d\theta)^2,$$
$$(dy)^2 = ((r')^2\sin^2(\theta) + 2rr'\sin(\theta)\cos(\theta) + r^2\cos^2(\theta))(d\theta)^2.$$

Summing down columns gives,

$$(dx)^2 + (dy)^2 = [(r')^2 + r^2](d\theta)^2.$$

Taking square roots gives the differential element of arc length for a polar curve,

$$ds = \boxed{\sqrt{[r'(\theta)]^2 + [r(\theta)]^2}d\theta.}$$

Example. For the cardioid,

$$r(\theta) = a(1 + \cos(\theta)),$$

the derivative is,

$$r'(\theta) = -a\sin(\theta).$$

Thus,

$$(r')^2 + r^2 = a^2(1+\cos(\theta))^2 + (-a\sin(\theta))^2 = a^2(1 + 2\cos(\theta) + \cos^2(\theta)) + a^2\sin^2(\theta).$$

This simplifies to,

$$2a^2(1 + \cos(\theta)).$$

To simplify this further, write $\theta = 2(\theta/2)$ and use the double-angle formula to get,

$$2a^2(1+\cos(2(\theta/2))) = 2a^2(1+\cos^2(\theta/2) - \sin^2(\theta/2)) = 2a^2(2\cos^2(\theta/2)) = 4a^2\cos^2(\theta/2).$$

Taking square roots gives,

$$ds = \boxed{2a\cos(\theta/2).}$$

Note, this answer is only correct for $-\pi \leq \theta \leq \pi$. Outside this range, we might have to take the other square root to get a positive number. In particular, the total arc length of the cardioid is,

$$s = \int ds = \int_{\theta=-\pi}^{\theta=\pi} 2a\cos(\theta/2)d\theta = 2a\left(2\sin(\theta/2)\right|_{-\pi}^{\pi} = 2a((2) - (-2)).$$

Simplifying, the total arc length of the cardioid is,

$$s = \boxed{8a.}$$

Surface areas of surfaces of revolution can be computed in a similar way. This was only briefly discussed in lecture. Here is a continuation of the previous problem.

Example. The top half of the cardioid,

$$r(\theta) = a(1 + \cos(\theta)), \quad 0 \leq \theta \leq \pi,$$

is revolved about the x-axis to give a fairly good approximation of the surface of an apple. What is the surface area of this apple?

Since we are revolving about the x-axis, the radius of each slice is y. Therefore the differential element of surface area is,

$$dA = 2\pi y ds.$$

Substituting in $y = r(\theta)\sin(\theta) = a(1+\cos(\theta))\sin(\theta)$, and substituting in for ds gives,

$$dA = 2\pi[a(1+\cos(\theta))\sin(\theta)](2a\cos(\theta/2)d\theta).$$

To simplify this, substitute both,

$$1+\cos(\theta) = 2\cos^2(\theta/2),$$

and,

$$\sin(\theta) = 2\sin(\theta/2)\cos(\theta/2),$$

to get,

$$dA = 4\pi a^2(2\cos^2(\theta/2))(2\sin(\theta/2)\cos(\theta/2))\cos(\theta/2)d\theta = 16\pi a^2\cos^4(\theta/2)\sin(\theta/2)d\theta.$$

Thus the total surface area is,

$$A = \int dA = \int_{\theta=0}^{\pi} 16\pi a^2\cos^4(\theta/2)\sin(\theta/2)d\theta.$$

To evaluate this integral, substitute,

$$\left.\begin{array}{r} u = \cos(\theta/2) \\ du = -(1/2)\sin(\theta/2)d\theta, \end{array}\right| \begin{array}{l} u(\pi) = 0, \\ u(0) = 1 \end{array}$$

The new integral is,

$$A = 16\pi a^2\int_{u=1}^{u=0} u^4(-2du) = 32\pi a^2\int_{u=0}^{u=1} u^4 du = 32\pi a^2\left(\frac{u^5}{5}\right|_0^1.$$

This evaluates to give the total surface area of the apple,

$$A = \boxed{32\pi a^2/5.}$$

5. Area of a region enclosed by a polar curve. What is the area of the planar region enclosed by a cardioid? By the same sort of reasoning as for volumes and arc lengths, the differential element of area of the triangular region bounded by the rays θ, $\theta + d\theta$ and the curve $r(\theta)$ is,

$$dA = \frac{r(\theta)^2}{2}d\theta.$$

Thus the area enclosed by a polar curve is,

$$A = \int dA = \int_{\theta=a}^{\theta=b}\frac{r(\theta)^2}{2}d\theta.$$

In particular, the area enclosed by the cardioid is,

$$A = \int_0^{2\pi} \frac{a^2(1+\cos(\theta))^2}{2} d\theta.$$

This expands to give,

$$\frac{a^2}{2} \int_0^{2\pi} 1 + 2\cos(\theta) + \cos(\theta)^2 d\theta.$$

To simplify the last part of the integrand, substitute,

$$\cos(\theta)^2 = \frac{1+\cos(2\theta)}{2},$$

to get,

$$\frac{a^2}{2} \int_0^{2\pi} 1 + 2\cos(\theta) + \frac{1+\cos(2\theta)}{2} d\theta = \frac{a^2}{4} \int_0^{2\pi} 3 + 4\cos(\theta) + \cos(2\theta) d\theta.$$

Using the Fundamental Theorem of Calculus, this equals,

$$\frac{a^2}{4} \left(3\theta + 4\sin(\theta) + \frac{1}{2}\sin(2\theta) \right|_0^{2\pi}.$$

Evaluating gives,

$$A = \boxed{3\pi a^2/2.}$$