

Lecture 21. November 3, 2005

Homework. Problem Set 6 Part I: (a) - (e); Part II: Problem 1.

Practice Problems. Course Reader: 4E-2, 4E-5, 4E-7, 4F-1, 4F-6.

1. Parametric equations. To this point in the course, plane curves were specified in 1 of 2 ways. The *explicit form*, or *graph form* of a curve in Cartesian coordinates is the common form,

$$y = f(x), \quad a \leq x \leq b.$$

The *implicit form* of a curve in Cartesian coordinates is as the set of all solutions of an equation,

$$F(x, y) = 0.$$

Often a subset of this curve is specified by imposing extra conditions, e.g., the upper unit semicircle is the set of solutions of $x^2 + y^2 = 1$ satisfying the extra condition $y > 0$.

There is a third important way to specify a curve: using *parametric equations*. Given a *parameter* t varying in an interval $a \leq t \leq b$ and given functions $f(t)$ and $g(t)$ on this interval, the associated parametric curve,

$$\begin{cases} x = f(t), \\ y = g(t) \end{cases}$$

is simply the set of all pairs $(x, y) = (f(t), g(t))$ as t varies over the interval $a \leq t \leq b$. We consider only the case where $f(t)$ and $g(t)$ are piecewise differentiable functions (more advanced courses discuss some pitfalls if $f(t)$ and $g(t)$ are merely continuous functions).

Examples. A. One specification of the points on the circle of radius r centered at $(0, 0)$ is using the angle θ . This gives rise to a parametric equation with parameter θ ,

$$\begin{cases} x = r \cos(\theta), \\ y = r \sin(\theta) \end{cases} \quad 0 \leq \theta < 2\pi.$$

B. An ellipse centered at $(0, 0)$ whose axes equal the coordinate axes has a parametric equation,

$$\begin{cases} x = a \cos(\theta), \\ y = b \sin(\theta) \end{cases} \quad 0 \leq \theta < 2\pi.$$

C. A projectile is launched from an initial position of (x_0, y_0) with an initial velocity vector of magnitude v_0 at an angle α to the horizontal, and under the influence of constant gravitational acceleration $-g$. According to Newton's laws of mechanics, the position of the projectile after time t is,

$$\begin{cases} x = v_0 \cos(\alpha)t + x_0, \\ y = -(g/2)t^2 + v_0 \sin(\alpha)t + y_0 \end{cases} \quad 0 \leq t.$$

This is a parametric equation where time t is the parameter. Even when some other quantity is the parameter, it is often useful to think of the parameter as time. Thus the curve is the trail left by a point, or perhaps better, the tip of a pen, as it moves in the plane.

2. Implicitization. Under reasonable hypotheses, it is possible to turn a portion of an implicit curve into an explicit curve. Similarly, it should be possible to turn a portion of a parametric curve into an explicit curve. It is often simpler to find an implicit equation satisfied by a parametric curve. The process of finding an implicit equation is called *implicitization*.

Examples. A. For the parametric curve in Example A above, by the Pythagorean Theorem,

$$x(\theta)^2 + y(\theta)^2 = r^2 \cos^2(\theta) + r^2 \sin^2(\theta) = r^2(\cos^2(\theta) + \sin^2(\theta)) = r^2.$$

Thus the parametric equation satisfies the implicit equation,

$$x^2 + y^2 = r^2.$$

B. For the parametric curve in Example B,

$$(x(\theta)/a)^2 + (y(\theta)/b)^2 = \cos^2(\theta) + \sin^2(\theta) = 1.$$

Thus the parametric equation satisfies the implicit equation,

$$x^2/a^2 + y^2/b^2 = 1.$$

C. For the parametric curve in Example C, assuming $v_0 \cos(\alpha)$ is nonzero, the equation for x can be solved for t ,

$$x = v_0 \cos(\alpha)t + x_0 \Leftrightarrow t = \frac{x - x_0}{v_0 \cos(\alpha)}.$$

This can then be substituted into the equation for y to get an explicit equation for the curve,

$$y = -g(x - x_0)^2 / (2v_0^2 \cos^2(\alpha)) + \tan(\alpha)(x - x_0) + y_0.$$

In going from a parametric equation to an implicit equation, there are 2 important warnings to keep in mind:

- A parametric equation may traverse only part of the implicit curve. The most usual reason is that the parameter t is restricted to a certain range. A closely related reason is that the functions of t are themselves somehow limited, as in the parametric curve lying in the line $y = x$,

$$\begin{cases} x = \cos(t), \\ y = \cos(t) \end{cases}$$

A more interesting reason is that the implicit curve may have more than one connected piece, as in the parametric curve,

$$\begin{cases} x = 2t/(1-t^2), \\ y = (1+t^2)/(1-t^2) \end{cases} \quad -1 < t < 1.$$

As t varies, this parametric curve sweeps out the top branch of the hyperbola $y^2 - x^2 = 1$.

- A parametric equation may sweep out all or a portion of the implicit curve multiple times. This is clear in Examples A and B: as θ is allowed to vary over the interval $0 \leq \theta < 2n\pi$, the parametric curve completes n revolutions of the implicit curve.

3. Arc length. Given a segment of curve, what is the length of the curve? Imagining the curve made of some flexible extensible material like wire, what is the length when the wire is pulled taut? The answer is called the *arc length*, s .

The method for expressing arc length as an integral is by now familiar. Break the interval $a \leq t \leq b$ into a large number n of subintervals with endpoints,

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b.$$

Approximate the curve on each subinterval $t_{k-1} \leq t \leq t_k$ by a line segment. The line segment runs from the point,

$$(x_{k-1}, y_{k-1}) = (x(t_{k-1}), y(t_{k-1})),$$

to the point,

$$(x_k, y_k) = (x(t_k), y(t_k)).$$

The rise and run of the line segment are,

$$\Delta x_k = x_k - x_{k-1} \approx x'(t_k) \Delta t_k,$$

$$\Delta y_k = y_k - y_{k-1} \approx y'(t_k) \Delta t_k.$$

By the Pythagorean theorem, the length of the line segment is,

$$\Delta s_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} \approx \sqrt{(x'(t_k))^2 + (y'(t_k))^2} \Delta t_k.$$

The arc length of the curve is approximately the sum of the lengths of the approximating line segments,

$$s \approx \sum_{k=1}^n \sqrt{(x'(t_k))^2 + (y'(t_k))^2} \Delta t_k.$$

This is a Riemann sum. As the mesh of the partition tends to 0, the Riemann sums tend to a Riemann integral. This Riemann integral is the arc length,

$$\text{Arc length} = \int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

Example 1. For the parametric curve in Example A above,

$$\frac{dx}{d\theta} = -r \sin(\theta), \quad \frac{dy}{d\theta} = r \cos(\theta).$$

Therefore, the expression,

$$\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2,$$

equals,

$$(-r \sin(\theta))^2 + (r \cos(\theta))^2 = r^2 \sin^2(\theta) + r^2 \cos^2(\theta) = r^2(\sin^2(\theta) + \cos^2(\theta)) = r^2.$$

Taking square roots gives the equation,

$$\frac{ds}{d\theta} = r, \quad \Leftrightarrow ds = r d\theta.$$

Thus the arc length of the arc of the circle from $\theta = a$ to $\theta = b$ is,

$$s = \int ds = \int_{\theta=a}^{\theta=b} r d\theta = r(b-a).$$

This is, in fact, our definition of the angle: the angle θ subtended by an arc of a circle equals the ratio of the arc length by the radius of the circle. If this logic sounds circular, it is perhaps that nobody ever told you before how to **define** the length of the arc of a circle! It is also an argument in favor of the more natural definition of the angle as the ratio of the *area* of the sector of the circle by $r^2/2$.

Example 2. This is not a single example, but a class of examples. A curve in explicit form, $y = f(x)$, $a \leq x \leq b$, can always be put in parametric form,

$$\begin{cases} x = t, \\ y = f(t) \end{cases} \quad a \leq t \leq b.$$

Then,

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = f'(t).$$

Using this,

$$ds = \sqrt{1 + (f'(t))^2} dt.$$

Thus the arc length is,

$$\text{Arc length} = \int_{t=a}^{t=b} \sqrt{1 + (f'(t))^2} dt.$$

Since the parameter t in the Riemann integral is only a dummy variable anyway, it is allowed to replace it by the variable x (so long as x plays no other role in the integral, which it does not). This gives the formula for the arc length of an explicit curve,

$$\text{Arc length} = \int_a^b \sqrt{1 + (dy/dx)^2} dx.$$

Example 3. Consider the explicit curve,

$$y = \frac{x^2}{4} - \frac{1}{2} \ln(x), \quad a \leq x \leq b,$$

where a is a positive real number. The derivative is,

$$\frac{dy}{dx} = \frac{1}{4}(2x) - \frac{1}{2} \frac{1}{x} = \frac{1}{2}(x - x^{-1}).$$

Thus the square of the derivative is,

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{4}(x^2 - 2 + x^{-2}).$$

Clearing denominators, $1 + (dy/dx)^2$ equals,

$$\frac{4}{4} + \frac{x^2 - 2 + x^{-2}}{4} = \frac{x^2 + 2 + x^{-2}}{4}.$$

It is easy to check this equals the square,

$$\left(\frac{x + x^{-1}}{2}\right)^2.$$

This gives the formula,

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{1}{2}(x + x^{-1}).$$

Therefore,

$$ds = \frac{1}{2}(x + x^{-1})dx.$$

Integrating gives the arc length,

$$s = \int ds = \int_{x=a}^{x=b} \frac{1}{2}(x + x^{-1})dx = \frac{1}{2} \left(\frac{x^2}{2} + \ln(x) \right) \Big|_a^b.$$

Evaluating gives the arc length,

$$\text{Arc length} = (b^2 - a^2)/4 + (1/2) \ln(b/a).$$