Lecture 18. October 25, 2005

Homework. Problem Set 5 Part I: (c).

Practice Problems. Course Reader: 3G-1, 3G-2, 3G-4, 3G-5.

1. Approximating Riemann integrals. Often, there is no simpler expression for the antiderivative than the expression given by the Fundamental Theorem of Calculus. In such cases, the simplest method to compute a Riemann integral is to use the definition. However, this is not necessarily the most *efficient* method. Often trapezoids or segments under a parabola give a better approximation to the Riemann integral than do vertical strips.

2. The trapezoid rule. The problem is to find an approximation of the Riemann integral,

$$I = \int_a^b y dx$$

for a function $y(x)$ defined on the interval $[a, b]$. Choose a partition of the interval $[a, b]$ into n equal subintervals. The points of this partition are,

$$x_k = a + \frac{(b-a)k}{n}, \quad \Delta x_k = \frac{b-a}{n}.$$

The values of these points are,

$$y_k = f(x_k).$$

The Riemann sum using always the left endpoint is,

$$I_l = \sum_{k=1}^{n} y_{k-1} \Delta x_k.$$

The Riemann sum using always the right endpoint is,

$$I_r = \sum_{k=1}^{n} y_k \Delta x_k.$$

The average of the two is,

$$I_{\text{trap}} = \sum_{k=1}^{n} \frac{y_{k-1} + y_k}{2} \Delta x_k.$$

This is usually a better approximation than either of the two approximations individually. Part of the reason is that the term $(y_{k-1} + y_k)\Delta x_k/2$ is the area of the *trapezoid* containing the points $(x_{k-1}, 0)$, (x_{k-1}, y_{k-1}), $(x_k, 0)$ and (x_k, y_k). In particular, if the graph of $y = f(x)$ is a line, this trapezoid is precisely the region between the graph and the x-axis over the interval $[x_{k-1}, x_k]$. Thus, the approximation above gives the *exact* integral for linear integrands.

Writing out the sum gives,

$$I_{\text{trap}} = \frac{b-a}{2n}((y_0 + y_1) + (y_1 + y_2) + (y_2 + y_3) + \cdots + (y_{n-2} + y_{n-1}) + (y_{n-1} + y_n)).$$

Gathering like terms, this reduces to,

$$I_{\text{trap}} = \boxed{(b-a)(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n)/2n.}$$

3. Simpson's rule. Again partition the interval $[a, b]$ into n equal subintervals. For reasons that will become apparent, n must be even. So let $n = 2m$ where m is a positive integer. Again define,

$$x_k = a + \frac{(b-a)k}{n} = a + \frac{(b-a)k}{2m}, \ \Delta x_k = \frac{b-a}{n} = \frac{b-a}{2m}.$$

Pair off the intervals as $([x_0, x_1], [x_1, x_2])$, $([x_2, x_3], [x_3, x_4])$, etc. Thus the l^{th} pair of intervals is,

$$([x_{2l-2}, x_{2l-1}], [x_{2l-1}, x_{2l}]).$$

The idea is to approximate the area of the graph over the pair of intervals by the area under the unique parabola containing the 3 points $(x_{2l-2}, y_{2l-2}), (x_{2l-1}, y_{2l-1}), (x_{2l}, y_{2l})$. For notation's sake, denote $2l-1$ by k. Thus the 3 points are (x_{k-1}, y_{k-1}), (x_k, y_k), and (x_{k+1}, y_{k+1}) (this is slightly more symmetric).

The first problem is to find the equation of this parabola. Since the parabola contains the point (x_k, y_k), it has the equation,

$$y = A(x - x_k)^2 + B(x - x_k) + y_k,$$

Plugging in $x = x_{k-1}$ and $x = x_{k+1}$, and using that $x_{k+1} - x_k = x_k - x_{k-1}$ equals Δx,

$$y_{k+1} = A(\Delta x)^2 + B(\Delta x) + y_k,$$

$$y_{k-1} = A(\Delta x)^2 - B(\Delta x) + y_k.$$

Summing the two sides gives,

$$y_{k+1} + y_{k-1} = 2A(\Delta x)^2 + 2y_k.$$

Solving for A gives,

$$A = \frac{1}{2(\Delta x)^2}(y_{k-1} - 2y_k + y_{k+1}).$$

Similarly, taking the difference of the two sides gives,

$$y_{k+1} - y_{k-1} = 2B(\Delta x).$$

Solving for B gives,

$$B = \frac{1}{2(\Delta x)}(y_{k+1} - y_{k-1}).$$

Thus, the equation of the parabola passing through (x_{k-1}, y_{k-1}), (x_k, y_k) and (x_{k+1}, y_{k+1}) is,

$$y = A(x - x_k)^2 + B(x - x_k)^2 + y_k,$$

$$A = (y_{k-1} - 2y_k + y_{k+1})/2(\Delta x)^2,$$

$$B = (y_{k+1} - y_{k-1})/2(\Delta x).$$

The next problem is to compute the area under the parabola from $x = x_{k-1}$ to $x = x_{k+1}$. This is a straightforward application of the Fundamental Theorem of Calculus,

$$\int_{x_{k-1}}^{x_{k+1}} A(x - x_k)^2 + B(x - x_k) + y_k dx = \left(\frac{A}{3}(x - x_k)^3 + \frac{B}{2}(x - x_k)^2 + y_k(x - x_k)\right|_{x_{k-1}}^{x_{k+1}}.$$

Plugging in and using that $x_{k+1} - x_k = x_k - x_{k-1}$ equals Δx, this is,

$$\frac{2A}{3}(\Delta x)^3 + 2y_k(\Delta x).$$

Substituting in the formula for A and simplifying, this is,

$$\frac{\Delta x}{3}(y_{k-1} - 2y_k + y_{k+1}) + \frac{\Delta x}{3}(6y_k) = \frac{\Delta x}{3}(y_{k-1} + 4y_k + y_{k+1}).$$

Back-substituting $2l-1$ for k and $(b-a)/2m$ for Δx, the approximate area for the pair of intervals $[x_{2l-2}, x_{2l-2}]$ and $[x_{2l-1}, x_{2l}]$ is,

$$\Delta I_l = \frac{b-a}{6m}(y_{2l-2} + 4y_{2l-1} + y_{2l}).$$

Finally, summing this contribution over each choice of l gives the Simpson's rule approximation,

$$I_{\text{Simpson}} = \frac{b-a}{6m}\sum_{l=1}^{m}(y_{2l-2} + 4y_{2l-1} + y_{2l}).$$

Writing out the sum gives,

$$I_{\text{Simpson}} = \tfrac{b-a}{6m}((y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + (y_4 + 4y_5 + y_6) + \cdots + (y_{2m-4} + 4y_{2m-3} + y_{2m-2}) + (y_{2m-2} + 4y_{2m-1} + y_{2m})).$$

Gathering like terms, I_{Simpson} reduces to,

$$(b-a)(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + 2y_6 + \cdots + 4y_{2m-3} + 2y_{2m-2} + 4y_{2m-1} + y_{2m})/6m.$$

Example. Approximate ln(2) using a partition into 4 equal subintervals with the Trapezoid Rule and with Simpson's Rule.

The value ln(2) equals the Riemann integral,

$$\int_1^2 \frac{1}{x}dx.$$

The points of the partition are $x_0 = 4/4, x_1 = 5/4, x_2 = 6/4, x_3 = 7/4$ and $x_4 = 8/4$. The corresponding values are $y_0 = 4/4, y_1 = 4/5, y_2 = 4/6, y_3 = 4/7, y_4 = 4/8$. Thus the Trapezoid Rule gives,

$$I_{\text{trap}} = \frac{b-a}{2n}(y_0 + 2y_1 + 2y_2 + 2y_3 + y_4) = \frac{1}{8}(\frac{4}{4} + 2\frac{4}{5} + 2\frac{4}{6} + 2\frac{4}{7} + \frac{4}{8}) = \frac{1171}{1680} \approx \boxed{0.6970}$$

For Simpson's Rule, because n equals 4, m equals 2. Thus,

$$I_{\text{Simpson}} = \frac{b-a}{6m}(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4) = \frac{1}{12}(\frac{4}{4} + 4\frac{4}{5} + 2\frac{4}{6} + 4\frac{4}{7} + \frac{4}{8}) = \frac{1747}{2520} \approx \boxed{0.6933}$$

According to a calculator, the true value is,

$$\ln(2) = \boxed{0.6931 \pm 10^{-4}}$$

Note that trapezoids overestimate the area, because $1/x$ is concave up. The approximating parabolas cross the graph of $y = 1/x$, thus the underestimation to the left of (x_k, y_k) somewhat cancels the overestimation to the right of (x_k, y_k), explaining the better approximation.

4. One review problem. This is a related rates review problem for Exam 3. A particle moves with constant speed 3 on the parabola $y = x^2$. The particle is moving away from the origin. What is the rate-of-change of the distance from the origin to the particle when the distance equals $2\sqrt{5}$?

The independent variable is time, t. The dependent variables are the x-coordinate of the particle, $x(t)$, the y-coordinate of the particle, $y(t)$, and the distance $L(t)$ from the particle to $(0, 0)$. The constant is the speed $s = 3$ of the particle. The constraints are that the point moves on the parabola,

$$y = x^2,$$

and the Pythagorean theorem,

$$L^2 = x^2 + y^2.$$

Also, since the speed is constant,

$$s^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2.$$

This plays the role of the "known rate-of-change" in a typical related rates problem.

It is simplest to relate the dependent variables y and L to x. The first step is to determine x at the moment when L equals $2\sqrt{5}$. Plugging $y = x^2$ into the equation for L^2 gives,

$$L^2 = x^2 + y^2 = x^2 + (x^2)^2 = x^2 + x^4.$$

At the instant when L equals $2\sqrt{5}$, L^2 equals 20. Thus, at that moment,

$$x^4 + x^2 = 20.$$

This factors as,

$$(x^2 - 4)(x^2 + 5) = 0.$$

Since x^2 is nonnegative, the solution is $x^2 = 4$. Assuming the particle is in the first quadrant (this is not specified in the problem), x is positive. The other choice leads to a symmetric problem and the same final answer. So, at the moment when L equals $2\sqrt{5}$, x equals 2.

The next step is to determine the "known rate-of-change", dx/dt at the moment when L equals $2\sqrt{5}$. Implicitly differentiating the equation $y = x^2$ gives,

$$\frac{dy}{dt} = 2x\frac{dx}{dt}.$$

Substituting this into the equation for s^2 gives,

$$s^2 = \left(\frac{dx}{dt}\right)^2 + \left(2x\frac{dx}{dt}\right)^2 = (1+4x^2)\left(\frac{dx}{dt}\right)^2.$$

Since s is known to be 3, and x is known to be 2, this equation can be solved for dx/dt,

$$\left(\frac{dx}{dt}\right)^2 = \frac{3^2}{1+4(2)^2} = \frac{9}{17}.$$

Since the particle is in the first quadrant and moving *away* from the origin, dx/dt is positive. So, at the moment when L equals $2\sqrt{5}$, dx/dt equals $3/\sqrt{17}$.

The final step is to compute dL/dt at the moment when L equals $2\sqrt{5}$. Implicitly differentiating the equation,

$$L^2 = x^2 + x^4,$$

gives,

$$2L\frac{dL}{dt} = (2x+4x^3)\frac{dx}{dt}.$$

Plugging in for L, x and dx/dt gives,

$$2(2\sqrt{5})\frac{dL}{dt} = (2(2)+4(2)^3)\frac{3}{\sqrt{17}}.$$

Solving gives,

$$\frac{dL}{dt} = \boxed{27/\sqrt{85}}.$$

at the moment when L equals $2\sqrt{5}$.