Lecture 17. October 21, 2005

Homework. Problem Set 5 Part I: (a) and (b); Part II: Problem 1.

Practice Problems. Course Reader: 3F-1, 3F-2, 3F-4, 3F-8.

1. Ordinary differential equations. An ordinary differential equation is an equation involving a single independent variable x together with a dependent variable y and its derivatives $d^k y/dx^k$,

$$G\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^ky}{dx^k}\right) = 0.$$

The largest k for which $d^k y/dx^k$ occurs in the equation is called *order* of the differential equation.

Examples. Here are examples of ordinary differential equations.

(i) The ordinary differential equation,

$$y - \sin(x^2) = 0,$$

has order 0, because no derivatives of y actually occur in the equation. It has a unique (and rather trivial) solution,

$$y = \sin(x^2).$$

Because the solution is unique, it depends on 0 parameters (and the order is 0).

(ii) The ordinary differential equation,

$$\frac{dy}{dx} - \frac{1}{x+1} = 0,$$

has order 1 because dy/dx occurs and no higher derivatives occur. Every solution is an antiderivative of $1/x + 1$,

$$y = \int \frac{1}{x+1} dx = \ln(|x+1|) + C,$$

Notice the solution depends on 1 parameter, C. And the order is 1.

(iii) The ordinary differential equation,

$$\frac{d^2y}{dx^2} + \omega^2 y = 0,$$

has order 2. The general solution was found in Problem Set 2, Problem 4,

$$y = A\cos(\omega x) + B\sin(\omega x).$$

The solution depends on 2 parameters, A and B. And the order is 2.

(iv) The previous equation was one particular linear ordinary differential equation. A k^{th} *order linear ordinary differential equation* has the form,

$$a_k(x)\frac{d^k y}{dx^k} + a_{k-1}(x)\frac{d^{k-1}y}{dx^{k-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = b(x),$$

for functions $a_k(x), \dots, a_0(x), b(x)$. If $b(x)$ is zero, the equation is *homogeneous*. Otherwise it is *inhomogeneous*. Very important is the case when all the functions $a_k(x), \dots, a_0(x), b(x)$ are constant. Then the differential equation is called *constant coefficient*. The solution of constant coefficient linear ordinary differential equations is a main focus of Math 18.03.

2. Separable differential equations. Many differential equations arising in applications are examples of separable differential equation. A *separable ordinary differential equation* is a first-order differential equation,

$$\frac{dy}{dx} = F(x, y),$$

for which $f(x, y)$ factors as,

$$F(x, y) = g(x)/h(y).$$

Example. Find the equation $y = f(x)$ of every curve with the following property: For every point (x, y) on the curve, the tangent line to the curve is perpendicular to the line joining (x, y) to the origin $(0, 0)$.

The slope of the tangent line to the curve at (x, y) is dy/dx. The slope of the line joining $(0, 0)$ and (x, y) is y/x. Since the tangent line is perpendicular to the line joining $(0, 0)$ and (x, y),

$$\frac{dy}{dx} = -x/y.$$

Thus, the equation $y = f(x)$ is a solution to this separable differential equation.

The algorithm for solving a separable differential equation is the following.

(i). Factor $f(x, y)$ as $g(x)/h(y)$. This is often the most difficult step. In the example, it is quite easy. Simply take $g(x) = -x$ and $h(y) = y$.

(ii). Rewrite the differential equation as an equality of differentials. In other words, rewrite the equation as,

$$\frac{dy}{dx} = \frac{g(x)}{h(y)} \Rightarrow h(y)dy = g(x)dx.$$

In the example, this gives,

$$\frac{dy}{dx} = \frac{-x}{y} \Rightarrow ydy = -xdx.$$

(iii). Antidifferentiate both sides of the equation. In the example, the antiderivatives

$$\int ydy = \int -xdx,$$

give,

$$\frac{1}{2}y^2 = \frac{-1}{2}x^2 + C.$$

(iv). If there is an inital value, use it to find the constant of integration. An *initial value problem* is an ordinary differential equation together with some information for an initial value x_0 of the independent variable. It is often written,

$$\begin{cases} dy/dx = F(x, y), \\ y(x_0) = y_0. \end{cases}$$

The example was not an initial value problem. However, it can easily be made an initial value problem by specifying,

$$y(1) = 1,$$

for instance. With this condition, the constant C satisfies the equation,

$$\frac{1}{2}(1)^2 = \frac{-1}{2}(1)^2 + C.$$

The solution is,

$$C = 1.$$

(v). Simplify the answer. Often it is best to solve for $y = f(x)$. Often this is unnecessary. It depends on the problem. In the example problem, the simplest answer is the implicit answer,

$$\boxed{x^2 + y^2 = 2C.}$$

So the solution of the initial value problem is,

$$\boxed{x^2 + y^2 = 2.}$$

Thus every curve satisfying the geometric property is a circle centered at the origin.

Example. Here is a somewhat different example. There is a single separable ordinary differential equation satisfied by every function,

$$y = (x - a)^3,$$

where a is an arbitrary constant. Find this differential equation, and find all its solutions.

The derivative of y is,

$$\frac{dy}{dx} = 3(x-a)^2.$$

The constant a can be eliminated by writing this as,

$$\frac{dy}{dx} = 3[(x-a)^3]^{2/3} = 3y^{2/3}.$$

This is a separable differential equation,

$$\boxed{dy/dx = 3y^{2/3}.}$$

The algorithm gives,

$$\begin{aligned} y^{-2/3}dy &= 3dx, \\ \int y^{-2/3}dy &= \int 3dx, \\ 3y^{1/3} &= 3x + C. \end{aligned}$$

Calling the constant $-3a$ gives the answer,

$$y = (x-a)^3.$$

However, there are **other** solutions. For instance, $y = 0$ is a solution. The general solution of the differential equation depends on 2 parameters, $a < b$,

$$y = \begin{cases} (x-a)^3, & x \leq a, \\ 0, & a < x \leq b, \\ (x-b)^3, & x > b \end{cases}$$

The problem is that in the step giving $dy/y^{2/3} = dx$. If y equals 0, this equation involves division by zero. Division by zero is not allowed, so the method breaks down.

Important fact. This fact will not be used in this class. However, it is often crucial in real-world applications to know the solution to an initial value problem is unique. The fact is,

$$\begin{cases} \frac{dy}{dx} = F(x,y), \\ y(x_0) = y_0, \end{cases}$$

has a unique solution for x close to x_0 if $F(x,y)$ is both continuous and differentiable at (x_0, y_0). In the previous example, $F(x,y) = 3y^{2/3}$ is continuous at $y_0 = 0$. But it is not differentiable at $y_0 = 0$. Ultimately, this is the reason for the extra solutions of the differential equation.

3. Applications. Separable differential equations come up often in applications. The most common separable differential equation is the equation for *exponential growth*,

$$\frac{dy}{dt} = ky,$$

where k is a constant.

The solution behaves differently if k is positive or negative. For k positive, this equation arises in population growth and interest on savings, among others. For k negative, this equation arises in radioactive decay, a discharging capactior in an RC-circuit, and Newton's law of cooling.

Population growth. The simplest model of population growth is that a population $N(t)$ (modeled as continuous for simplicity) grows at a rate proportional to the size of the population. This gives,

$$\frac{dN}{dt} = kN.$$

Following the method gives,

$$\begin{array}{c} dN/N = kdt, \\ \int 1/N dN = \int kdt, \\ \ln(|N|) = kt + C. \end{array}$$

Exponentiating both sides gives,

$$N(t) = \boxed{N_0 e^{kt}.}$$

Observe that $N(t)$ increases without bound as t increases. When N is very large, the ecosystem cannot support such a population. Thus the model is only valid if $N(t)$ is not too large.

A slightly more realistic model hypothesizes a constant, equilibrium population N_{equi} sustainable indefinitely. The model is that the population grows at a rate proportional both to the population N and the difference $N_{\text{equi}} - N$,

$$\frac{dN}{dt} = kN(N_{\text{equi}} - N).$$

This is again a separable differential equation. It gives the solution,

$$N(t) = \boxed{N_0 N_{\text{equi}}/(N_0 + (N_{\text{equi}} - N_0)e^{-kN_{\text{equi}}t}).}$$

The most important feature is that $N(t)$ approaches N_{equi} as t increases. This is called the *steady-state solution*. In general, to find the steady-state solution to a separable ordinary differential equation, assume the solution is constant $y = y_1$ so that dy/dt is 0. In the original model of population growth, the only steady-state solution is $N = 0$. In the new model, there are 2 steady-state solutions, $N = 0$ and $N = N_{\text{equi}}$. In Math 18.03, stability is defined, and a method is given to show the only stable steady-state solution is $N = N_{\text{equi}}$.

Radioactive decay. A radioactive isotope decays to a more stable isotope at a rate proportional to the remaining radioactive isotope. Thus the mass $m(t)$ satisfies a differential equation,

$$\frac{dm}{dt} = -km.$$

Using the method, the solution is,

$$m(t) = \boxed{m_0 e^{-kt}.}$$

An important feature in decay problems is the half-life. The *half-life* is the length of time necessary for the mass of radioactive isotope to decrease to one-half the initial mass,

$$m(T_{\text{half}}) = m_0/2.$$

Solving in the formula gives,

$$T_{\text{half}} = \boxed{\ln(2)/k.}$$

Example. The half-life of a certain radioactive isotope is 20 years. How long is required for the mass to decrease to 1% of the initial mass? Using the formula above, $k = \ln(2)/25$. Therefore the equation for the mass is,

$$m(t) = m_0 e^{-\ln(2)t/25}.$$

Thus the time t_f when the mass equals $0.01m_0$ satisfies,

$$m_0 e^{-\ln(2)t_f/25} = m_0/100,$$

or,

$$\ln(2)t_f/25 = \ln(100) = 2\ln(10).$$

Solving gives,

$$t_f = 50\ln(10)/\ln(2) = \boxed{166 \text{ years.}}$$

Newton's Law of Cooling. Isaac Newton proposed a law for the rate-of-change of the temperature T of an object placed in a large, effectively infinite, environment at a fixed ambient temperature T_{amb}. The law is that the rate-of-change of T is negatively proportional to the temperature gradient $T - T_{\text{amb}}$,

$$\frac{dT}{dt} = -k(T - T_{\text{amb}}).$$

The method gives the solution,

$$T(t) = T_{\text{amb}} + (T - T_{\text{amb}})e^{-kt}.$$

As t increases, the temperature T approaches the steady-state temperature, T_{amb}.