**Lecture 13.** October 13, 2005

**Homework.** Problem Set 4 Part I: (a) and (b); Part II: Problem 3.

**Practice Problems.** Course Reader: 3A-1, 3A-2, 3A-3.

**1. Differentials.** An alternative notation for derivatives is *differential notation.* The differential notation,

$$dF(x) = f(x)dx,$$

is shorthand for the sentence "The derivative of $F(x)$ with respect to $x$ equals $f(x)$." Formally, this is related to the Leibniz notation for the derivative,

$$\frac{dF}{dx}(x) = f(x),$$

which means the same thing as the differential notation. It may look like the first and second equation are obtained by dividing and multiplying by the quantity $dx$. It is crucial to remember that $dF/dx$ is **not a fraction**, although the notation suggests otherwise.

In differential notation, some derivative rules have a very simple form, and are thus easier to remember. Here are a few derivative rules in differential notation.

$$\begin{array}{ccc} dF(x) & = & F'(x)dx \\ d(F(x)+G(x)) & = & dF(x)+dG(x) \\ d(cF(x)) & = & cdF(x) \\ d(F(x)G(x)) & = & G(x)dF(x)+F(x)dG(x) \\ d(F(x)/G(x)) & = & 1/(G(x))^2(G(x)dF(x)-F(x)dG(x)) \end{array}$$

The chain rule has a particularly simple form,

$$d(F(u)) = \frac{dF}{du}du = \frac{dF}{du}\frac{du}{dx}dx.$$

**Example.** Using differential notation, the derivative of $\sin(\sqrt{x^2+1})$ is,

$$d\sin((x^2+1)^{1/2}) = \cos((x^2+1)^{1/2})d(x^2+1)^{1/2} = \cos((x^2+1)^{1/2})(\tfrac{1}{2}(x^2+1)^{-1/2})d(x^2+1) = \cos((x^2+1)^{1/2})\tfrac{1}{2}(x^2+1)^{-1/2}(2xdx) = \boxed{x(x^2+1)^{-1/2}\cos((x^2+1)^{1/2})dx}.$$

**2. Antidifferentiation.** Recall, the basic problem of differentiation is the following.

**Problem (Differentiation).** Given a function $F(x)$, find the function $f(x)$ satisfying $\frac{dF}{dx} = f(x)$.

The bais problem of *antidifferentiation* is the inverse problem.

**Problem (Antidifferentiation).** Given a function $f(x)$, find a function $F(x)$ satisfying $\frac{dF}{dx} = f(x)$.

A function $F(x)$ solving the problem is called an *antiderivative of* $f(x)$, or sometimes an *indefinite integral of* $f(x)$. The notation for this is,

$$F(x) = \int f(x)dx.$$

The expression $f(x)$ is called the *integrand*. It is important to note, if $F(x)$ is one antiderivative of $f(x)$, then for each constant $C$, $F(x) + C$ is also an antiderivative of $f(x)$. The constant $C$ is called a *constant of integration.*

In a sense that can be made precise, the problem of differentiation has a complete solution whenever $F(x)$ is a "simple expression", i.e., a function built from the differentiable functions we have seen so far. Unfortunately, for very many simple functions $f(x)$, no antiderivative of $f(x)$ has a simple expression. In large part, this is what makes antidifferentiation difficult. Luckily, many of the most important simple functions $f(x)$ do have an antiderivative with a simple expression. One goal of this unit is to learn how to recognize when a simple antiderivative exists, and some tools to compute the antiderivative.

**3. Antidifferentiation. Guess-and-check.** The main technique for antidifferentiation is educated guessing.

**Example.** Find an antiderivative of $f(x) = x^2 + 2x + 1$. Since the derivative of $x^n$ is $nx^{n-1}$, it is reasonable to guess there is an antiderivative of the form $F(x) = Ax^3 + Bx^2 + Cx$. Differentiation gives,

$$\frac{dF}{dx} = 3Ax^2 + 2Bx + C.$$

Thus, $F(x)$ is an antiderivative of $f(x)$ if and only if,

$$3A = 1, \quad 2B = 2, \quad \text{and } C = 1.$$

This gives an antiderivative,

$$\int (x^2 + 2x + 1)dx = \boxed{\tfrac{1}{3}x^3 + x^2 + x + E},$$

where $E$ is any constant.

Guess-and-check is a game we can lose, as well as win. However, there are a few rules that better the odds in this guessing game. In fact, they are basically the same rules for derivatives in differential notation, simply written backwards.

$$\begin{array}{ccc} \int (f(x)+g(x))dx & = & \int f(x)dx + \int g(x)dx \\ \int cf(x)dx & = & c\int f(x)dx \\ \int f(u(x))u'(x)dx & = & \int f(u)du \end{array}$$

**4. Antidifferentiation. Integration by substitution.** The last rule above is very important, and called *integration by substitution.*

**Example.** Find an antiderivative of $x\sin(x^2)$. This time guess-and-check is much less effective. By roughly the same logic in the last example, we might guess an antiderivative has the form $Ax^3\sin(x^2)$. The derivative is $3Ax^2\sin(x^2) + 2Ax^4\cos(x^2)$. The first term is good, but the second term is bad. We can try to correct our guess by adding a term, $Ax^3\sin(x^2) - 2/5Ax^5\cos(x^2)$, whose derivative is now $3Ax^2\sin(x^2) + 4/5Ax^6\sin(x^2)$. This still doesn't work, and is leading in the wrong direction.

A better solution is to use integration by substitution. Observe part of $f(x)$ can be written as a function of $u(x) = x^2$. Also, the derivative $u'(x) = 2x$ occurs in $f(x)$ through $x = 1/2(2x) = u'(x)/2$. Thus,

$$x\sin(x^2) = sin(u(x))u'(x)/2, \quad u(x) = x^2.$$

Applying integration by substitution,

$$\begin{array}{c} \int x\sin(x^2)dx = \int \sin(u(x))\frac{1}{2}u'(x)dx = \int \frac{1}{2}\sin(u)du = \\ \frac{-1}{2}\cos(u) + C = \boxed{\frac{-1}{2}\cos(x^2) + C.} \end{array}$$

Here is a checklist for applying integration by substition to find the antiderivative of $f(x)$.

(i) Find an expression $u(x)$ so that most of the integrand $f(x)$ can be expressed as a simpler function of $u(x)$.

(ii) Compute the differential $du(x) = u'(x)dx$.

(iii) Inside the differential $f(x)dx$, try to find $du = u'(x)dx$ as a factor.

(iv) Try to write $f(x)dx$ as $g(u)du$. If you cannot do this, the method does not apply with the given choice of $u$.

(v) Find an antiderivative $G(u) = \int g(u)du$ for the simpler integrand $g(u)$ (if this is possible).

(vi) Back-substitute $u = u(x)$ to get an antiderivative $F(x) = G(u(x))$ for $f(x)$.

**Example.** Compute the antiderivative,

$$\int \sin(x)^3 \cos(x) dx.$$

Most of the integrand is a function of $\sin(x)$. So substitute $u(x) = \sin(x)$. The differential of $u$ is $du = \cos(x)dx$. The differential $\sin(x)^3 \cos(x)dx$ contains $du = \cos(x)dx$ as a factor. The remainder of the integrand is $\sin(x)^3 = u^3$. So, according to integration by substitution,

$$\int \sin(x)^3 \cos(x) dx = \int u^3 du = \frac{1}{4}u^4 + C.$$

Finally, back-substitute $u = \sin(x)$ to get,

$$\int \sin(x)^3 \cos(x) dx = \boxed{(\sin(x))^4/4 + C.}$$