Lecture 11. October 4, 2005

Homework. Problem Set 3 Part I: (g) and (h).

Practice Problems. Course Reader: 2E-4, 2E-8, 2E-9.

1. Related rates. A situation that arises often in practice is that two quantities, say x and y, depend on a third independent variable, say t. The quantities x and y are related through some constraint. Using the constraint, if the rate-of-change dx/dt is known, the rate-of-change dy/dt can be inferred.

Example. For a spring displaced x units from equilibrium, Hooke's law implies the potential energy of the spring is,

$$P = \frac{1}{2}kx^2,$$

where k is a constant with units kg/s^2. At some moment $t = T$, a spring is displaced $5cm$ from equilibrium and has velocity $5cm/s$. In terms of the spring constant k, describe the rate-of-change of the potential energy at $t = T$.

Implicitly differentiating the equation with respect to t gives, using the chain rule,

$$\frac{dP}{dt} = \frac{1}{2}k(2x)\frac{dx}{dt} = kx\frac{dx}{dt}.$$

So, at time $t = T$,

$$\frac{dP}{dt}(T) = kx(T)\frac{dx}{dt}(T) = k(5)(5)cm^2/s = \boxed{25kcm^2/s.}$$

2. Method for solving related-rates problems. Many of these steps apply to any word-problem in mathematics.

(i) Identify the independent variable. In the example, this is t.

(ii) Label all constants. In the example, k is a constant.

(iii) Label all dependent variables. In the example, x and P are dependent variables.

(iv) Draw a diagram and carefully label it.

(v) Write the given rate-of-change and the unknown rate-of-change. In the example, $dx/dt(T)$ is given as $5cm/s$, and dP/dt is unknown.

(vi) Using the diagram and any other information, find constraints among the dependent variables. In the example, this is the equation $P = kx^2/2$.

(vii) Implicitly differentiate the constraint equations with respect to the independent variable. In the example, this gives $dP/dt = kxdx/dt$.

(viii) Substitute in all known quantities and solve for the unknown rate-of-change. In the example, $dP/dt(T)$ equals $25kcm^2/s$.

Example. A state trooper waits a distance a from a highway for passing speeders. The speed limit is $60mph$. The trooper aims her radar gun at an angle of $\pi/4$ to the road. The radar registers a passing car moving away from the trooper at a speed of $50mph$. Should the trooper ticket the driver?

The independent variable is time t. The constants are the distance a and the angle $\theta = \pi/4$. Label a coordinate system with the trooper at the origin and the highway equal to the line $y = a$. Label the position of the car along the highway as x, moving in the positive direction. Denote by r the distance of the car from the trooper. Then x and r are dependent variables. The rate-of-change $dr/dt(T)$ is given as $50mph$. The unknown rate-of-change is $dx/dt(T)$. The constraint is the Pythagorean theorem,

$$r^2 = x^2 + y^2.$$

Implicit differentiation with respect to t yields,

$$2r\frac{dr}{dt} = 2x\frac{dx}{dt} + 0 = 2x\frac{dx}{dt}.$$

At time $t = T$, $x(T)$ equals a, because the angle θ is $\pi/4$. Thus $r(T)$ equals $\sqrt{2}a$. Substituting in gives,

$$2(\sqrt{2}a)50 = 2(a)\frac{dx}{dt}(T).$$

Solving gives,

$$\frac{dx}{dt}(T) = \sqrt{2}50 \approx \boxed{71mph.}$$

So the trooper should ticket the driver.

Example. A point on the x-axis moves away from the origin. There is an angle θ subtended by the point and the unit circle with equation $x^2 + y^2 = 1$. In other words, standing at the point $(x, 0)$ and staring at the circle, θ is the angle of your field-of-vision occupied by the circle. At a moment $t = T$, the point is at the position $(2, 0)$ and moving with velocity v. What is the rate-of-change of θ at $t = T$?

The independent variable is time t. There is no constant. The dependent variables are the x-coordinate of the point, $x(t)$, and the angle $\theta(t)$. The rate-of-change $dx/dt(T)$ is given to be v. The rate-of-change $d\theta/dt$ is unknown.

The constraint is somewhat tricky. There are two tangent lines to the circle containing $(x, 0)$. These are the tangent lines to points $(a, +b)$ and $(a, -b)$ on the circle. Because the tangent line to the circle at (a, b) is perpendicular to the radius through (a, b), the triangle with vertices $(0, 0)$, (a, b) and the point $(x, 0)$ is a right triangle. The angle of the triangle at $(x, 0)$ is $\theta/2$. Since the radius has length 1 and the hypotenuse has length x, the constraint is,

$$\sin(\theta) = \frac{1}{x}.$$

Implicit differentiation with respect to t gives,

$$\frac{d\sin(\theta)}{d\theta}\frac{d\theta}{dt} = \frac{d(x^{-1})}{dx}\frac{dx}{dt},$$

or,

$$\cos(\theta)\frac{d\theta}{dt} = \frac{-1}{x^2}\frac{dx}{dt}.$$

Since $x(T)$ equals 2, $\sin(\theta(T)) = 1/2$, and thus $\cos(\theta(T))$ equals $\sqrt{3}/2$. Plugging in gives,

$$\frac{\sqrt{3}}{2}\frac{d\theta}{dt}(T) = \frac{-1}{(2)^2}v = \frac{-v}{4}.$$

Solving gives,

$$\frac{d\theta}{dt}(T) = \boxed{-v/(2\sqrt{3}).}$$

3. Another applied max/min problem. As review for Exam 2, this is another applied max/min problem. A trapezoid is inscribed inside the upper unit semicircle, $x^2 + y^2 = 1, y \geq 0$. The base of the trapezoid is the diameter of the semicircle lying on the x-axis. The top of the trapezoid is parallel to the x-axis joining $(-x, y)$ to (x, y) for a point (x, y) on the unit circle in the first quadrant. What is the maximal area enclosed by such a trapezoid?

The parameters are x and y. The height of the trapezoid is y. The area of a trapezoid is the product of the height with the average of the parallel sides. Thus,

$$A = y\frac{(2 + 2x)}{2} = (x + 1)y.$$

This is the quantity to be maximized. There is a constraint among the parameters,

$$x^2 + y^2 = 1.$$

Also, since (x, y) is in the first quadrant, $0 \leq x \leq 1$ and $0 \leq y \leq 1$.

There are at least 3 ways to proceed. The most direct is to solve for y in terms of x,

$$y = \sqrt{1 - x^2}.$$

Substituting this into the equation for A gives,

$$A(x) = (x + 1)\sqrt{1 - x^2}.$$

Differentiating gives,

$$\frac{dA}{dx} = \sqrt{1 - x^2} + (x + 1)\frac{-2x}{2\sqrt{1 - x^2}} = \frac{1}{\sqrt{1 - x^2}}((1 - x^2) - (x^2 + x)) = \frac{-(2x^2 + x - 1)}{\sqrt{1 - x^2}}.$$

Because the quadratic polynomial $2x^2 + x - 1$ factors as,

$$2x^2 + x - 1 = (2x - 1)(x + 1),$$

the critical points of A are $x = -1$ and $x = 1/2$. But $x = -1$ does not give a point in the first quadrant. Thus A is maximized either at one of the endpoints $x = 0$, $x = 1$ or at the critical point $x = 1/2$. Plugging in gives,

$$A(0) = 1, A(1/2) = 3\sqrt{3}/4, A(1) = 0.$$

This gives the answer,

$$\boxed{\text{A achieves its maximum } 3\sqrt{3}/4 \text{ for the point } (x, y) = (1/2, \sqrt{3}/2).}$$

Two other methods were given in lecture. The fastest among the three is to instead minimize A^2,

$$A^2 = (x+1)^2 y^2.$$

Using the constraint, $y^2 = 1 - x^2$, thus,

$$(A^2)(x) = (x+1)^2(1-x^2).$$

The derivative of this polynomial is very fast to compute, and gives the same answer as above.