

Lecture 10. September 30, 2005

Homework. Problem Set 3 Part I: (a)–(f). Part II: Problems 1, 2 and 3.

Practice Problems. Course Reader: 2C-5, 2C-10, 2C-12, 2D-3, 2D-4.

1. Asymptotes. An *asymptote* describes the behavior of the graph of $y = f(x)$ as it becomes unbounded, in some sense. There are two main examples. The function f has a *vertical asymptote* $x = a$ if at least 1 of the following holds,

$$\lim_{x \rightarrow a^-} f(x) = +\infty, \lim_{x \rightarrow a^-} f(x) = -\infty, \lim_{x \rightarrow a^+} f(x) = +\infty, \lim_{x \rightarrow a^+} f(x) = -\infty.$$

In each case, the graph of $y = f(x)$ becomes unbounded, and becomes arbitrarily close to the line $x = a$. If $x = a$ is a vertical asymptote, then $f(x)$ has an infinite discontinuity at $x = a$.

The function f has a *horizontal asymptote* $y = b$ if at least 1 of the following holds,

$$\lim_{x \rightarrow +\infty} f(x) = b, \lim_{x \rightarrow -\infty} f(x) = b.$$

In other words, the graph of $y = f(x)$ becomes arbitrarily close to the line $y = b$ as x approaches either $+\infty$ or $-\infty$.

Example. For the function $y = (x^3 + x)/(x^2 - 1) = x(x^2 + 1)/(x^2 - 1)$, the lines $x = -1$ and $x = 1$ are vertical asymptotes. There is no horizontal asymptote. However, the graph of y is *asymptotic* to the line $y = x$. This was not discussed in lecture. A pair of functions f and g are *asymptotic* to each other if the line $y = 0$ is a horizontal asymptote of $f - g$.

2. Applied maximum/minimum problems. Using the First Derivative Test, the maximum and minimum of many functions can be computed. This is very important in applications.

Example. Two long walls meet at right angles making a corner. Using a length of 10 meters of fence to form the other 2 sides of a rectangle, what is the largest area that can be enclosed in this corner?

Step 1. Identify parameters. A *parameter* is a constant or variable. The constant in this problem is 10 meters. Two variables are the length l of one side of the rectangle, and the width w of the remaining side of the rectangle.

Step 2. Draw a diagram. This was done in lecture.

Step 3. Find the quantity to be maximized or minimized. The quantity to be maximized is the area A of the rectangle. Since the area is the product of the length and width, A equals lw .

Step 4. Use the constraints to eliminate variables. The constraint is that the total length of fence is 10 meters. Thus $l + w$ equals 10. This is used to eliminate w ,

$$w = 10 - l.$$

Making this substitution, A is now a function of l alone,

$$A(l) = lw(l) = l(10 - l) = -l^2 + 10l.$$

Step 4 $\frac{1}{2}$. **Sketch a graph of the quantity to be maximized or minimized.** This is not absolutely necessary. Sometimes it is impossible. When you can make a rough sketch, this will typically give a very good idea where the maximum or minimum lies. In the example above, $A(l)$ is a quadratic equation. Because both l and w must be nonnegative, $A(l)$ is only defined on the interval $0 \leq l \leq 10$. Thus the graph of $A(l)$ is a segment of a parabola opening down. The vertex of the parabola is contained in the segment. Thus the vertex is the maximum.

Step 5. Compute the derivative. In this case,

$$A'(l) = -2l + 10.$$

Step 6. Find all critical points, endpoints, discontinuity points, etc. In most cases, it suffices to find all critical points and endpoints. Occasionally it is also necessary to find all points where f' is not defined. Rarely it is necessary to also consider discontinuity points (although this is usually so obvious that it does not require a separate step). In this case, the endpoints are $l = 0$ and $l = 10$. The one critical point is $l = 5$.

Step 7. Determine the global maximum or minimum. Checking all critical points, endpoints, etc., determine the global maximum or the global minimum. In this case, $A(0)$ equals 0, $A(10)$ equals 0 and $A(5)$ equals 25. Thus $l = 5$ is the global maximum.

Step 8. Back-substitute. Plug in the value of the single remaining independent variable to determine the values of the remaining independent variables. In this case, w equals $10 - l$, which is $10 - 5 = 5$ for $l = 5$. Thus, the largest area 25 is enclosed by a square of side length 5.

Example. A swimmer is in the water at a distance b_1 meters from shore. She wants to reach a point on land b_2 meters from the water. The point is a meters parallel to the shore. If the swimmer swims v_1 meters per second and runs v_2 meters per second, at what distance x from the closest point on shore should she aim to minimize her time to the target? Mathematically, the swimmer is at point $(0, b_1)$ and wants to reach point $(a, -b_2)$, where the shore is the x -axis. At what point $(x, 0)$ should she aim?

The constants are a , b_1 , b_2 , v_1 and v_2 . The variable is x . It is also convenient to introduce a variable d_1 for the distance from $(0, b_1)$ to $(x, 0)$, and a variable d_2 for the distance from $(x, 0)$ to $(a, -b_2)$. Although not obvious, it is also very convenient to introduce a variable θ_1 for the acute angle formed by the x -axis and the line segment joining $(0, b_1)$ to $(x, 0)$. Also introduce θ_2 for the acute angle formed by the x -axis and the line segment joining $(x, 0)$ to $(a, -b_2)$.

The time T_1 to swim to point $(x, 0)$ is,

$$T_1 = \frac{d_1}{v_1} = \frac{1}{v_1}(x^2 + b_1^2)^{1/2}.$$

The time T_2 to run from $(x, 0)$ to point $(a, -b_2)$ is,

$$T_2 = \frac{d_2}{v_2} = \frac{1}{v_2}((a - x)^2 + b_2^2)^{1/2}.$$

Thus the total time to reach the target is,

$$T = T_1 + T_2 = \frac{1}{v_1}(x^2 + b_1^2)^{1/2} + \frac{1}{v_2}((a-x)^2 + b_2^2)^{1/2}.$$

The derivative of T with respect to x is,

$$\frac{dT}{dx} = \frac{1}{v_1} \left(\frac{1}{2}(x^2 + b_1^2)^{-1/2}(2x) \right) + \frac{1}{v_2} \left(\frac{1}{2}((a-x)^2 + b_2^2)^{-1/2}(-2(a-x)) \right).$$

This simplifies to,

$$\frac{dT}{dx} = \frac{x}{v_1 d_1} - \frac{a-x}{v_2 d_2}.$$

Observe that x/d_1 equals $\sin(\theta_1)$ and $(a-x)/d_2$ equals $\sin(\theta_2)$. Thus,

$$\frac{dT}{dx} = \frac{\sin(\theta_1)}{v_1} - \frac{\sin(\theta_2)}{v_2}.$$

Technically, there are no endpoints. However, it is obvious that the maximum must occur for $0 \leq x \leq a$. Thus these may be taken to be endpoints. The critical value occurs when,

$$\frac{\sin(\theta_1)}{v_1} = \frac{\sin(\theta_2)}{v_2}.$$

This is *Snell's Law* for refraction of light upon crossing from one medium to another. For refraction, a particle of light (perhaps fictitious) replaces the swimmer, a translucent medium of one type replaces the water, and a translucent medium of a second type replaces the land. If light travels with velocity v_1 in the first medium and with velocity v_2 in the second medium, light rays will *refract* upon crossing the boundary between media. Snell's Law describes the angles of this refraction.