Lecture 1. September 8, 2005

Homework. Problem Set 1 Part I: (a)–(e); Part II: Problems 1 and 2.

Practice Problems. Course Reader: 1B-1, 1B-2
Textbook: p. 68, Problems 1–7 and 15.

1. Velocity. Displacement is $s(t)$. *Increment* from t_0 to $t_0 + \Delta t$ is,

$$\Delta s = s(t_0 + \Delta t) - s(t_0).$$

Average velocity from t_0 to $t_0 + \Delta t$ is,

$$v_{\text{ave}} = \frac{\Delta s}{\Delta t} = \frac{s(t_0 + \Delta t) - s(t_0)}{\Delta t}.$$

Velocity, or *instantaneous velocity*, at t_0 is,

$$v(t_0) = \lim_{\Delta t \to 0} v_{\text{ave}} = \lim_{\Delta t \to 0} \frac{s(t_0 + \Delta t) - s(t_0)}{\Delta t}.$$

This is a *derivative*, $v(t)$ equals $s'(t) = ds/dt$. The derivative of velocity is **acceleration**,

$$a(t_0) = v'(t_0) = \lim_{\Delta t \to 0} \frac{v(t_0 + \Delta t) - v(t_0)}{\Delta t}.$$

Example. For $s(t) = -5t^2 + 20t$, first computed velocity at $t = 1$ is,

$$v(1) = \lim_{\Delta t \to 0} 10 - 5\Delta t = \boxed{10.}$$

Then computed velocity at $t = t_0$ is,

$$v(t_0) = \lim_{\Delta t \to 0} -10t_0 + 10 - 5\Delta t = \boxed{-10t_0 + 20.}$$

Finally, computed acceleration at $t = t_0$ is,

$$a(t_0) = \lim_{\Delta t \to 0} -10 = \boxed{-10.}$$

2. Derivative. Let $y = f(x)$ be a *dependent variable* depending on an *independent variable* x, varying freely. The *increment* of y from x_0 to $x_0 + \Delta x$ is,

$$\Delta y = f(x_0 + \Delta x) - f(x_0).$$

The *difference quotient* or *average rate-of-change* of y from x_0 to $x_0 + \Delta x$ is,

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

The *derivative* of y (or $f(x)$) with respect to x at x_0 is,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

3. Examples in science and math.

(i) Economics. *Marginal cost* is the derivative of cost with respect to some other variable, for instance, the quantity purchased.

(ii) Thermodynamics. The ideal gas law relating pressure p, volume V, and temperature T of a gas is,

$$pV = nRT.$$

Under isothermal conditions, T is a constant T_0 so that,

$$p(V) = \frac{nRT_0}{V}.$$

Under adiabatic conditions (i.e., no transfer of heat), pV^γ is a constant K. Using this to eliminate p gives,

$$T(V) = \frac{K}{nR}\frac{1}{V^{\gamma-1}}.$$

As this illustrates, the independent variable, dependent variable and constants in an equation very much depend on the problem to be solved.

(iii) Biology. Exponential population growth models the population $N(t)$ after t years as,

$$N(t) = N_0 e^{rt},$$

where e^x is the exponential function, N_0 is initial population, and r is a growth factor. Later we will see, $N'(t) = rN(t)$, i.e., the population grows at a rate proportional to the size of the population.

(iv) Geometry. The volume of a right circular cone is,

$$V = \frac{1}{3} A \times h.$$

where A is the base area of the cone and h is the height of the cone. The radius r of the base is proportional to the height,

$$r(h) = ch,$$

for some constant c. Since $A = \pi r^2$, this gives,

$$V(h) = \frac{\pi}{3} c^2 h^3.$$

The derivative is,

$$\frac{dV}{dh} = \pi c^2 h^2 = \pi r^2 = \boxed{A.}$$

This is very reasonable. In some sense, this explains the classical formula for the volume of a cone.