18.01 Solutions to Exam 1

Problem 1(15 points) Use the definition of the derivative as a limit of difference quotients to compute the derivative of $y = x + \frac{1}{x}$ for all points $x > 0$. Show all work.

Solution to Problem 1 Denote by $f(x)$ the function $x + \frac{1}{x}$. By definition, the derivative of $f(x)$ at $x = a$ is,

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

The increment $f(a+h) - f(a)$ equals,

$$\left((a+h) + \frac{1}{a+h}\right) - \left(a + \frac{1}{a}\right) = h + \left(\frac{1}{a+h} - \frac{1}{a}\right).$$

To compute the second term, clear denominators,

$$\frac{1}{a+h} - \frac{1}{a} = \frac{1}{a+h}\frac{a}{a} - \frac{1}{a}\frac{a+h}{a+h} = \frac{a-(a+h)}{a(a+h)} = \frac{-h}{a(a+h)}.$$

Thus the increment $f(a+h) - f(a)$ equals,

$$h - \frac{h}{a(a+h)}.$$

Factoring h from each term, the difference quotient equals,

$$\frac{f(a+h) - f(a)}{h} = 1 - \frac{1}{a(a+h)}.$$

Thus the derivative of $f(x)$ at $x = a$ equals,

$$f'(a) = \lim_{h \to 0}\left(1 - \frac{1}{a(a+h)}\right) = 1 - \frac{1}{a(a+0)} = 1 - \frac{1}{a^2}.$$

Therefore the derivative function of $f(x)$ equals,

$$f'(x) = \boxed{1 - \frac{1}{x^2}.}$$

Problem 2(10 points) For the function $f(x) = e^{-x^2/2}$, compute the first, second and third derivatives of $f(x)$.

Solution to Problem 2 Set u equals $-x^2/2$ and set v equals e^u. So v equals $f(x)$. By the chain rule,

$$\frac{dv}{dx} = \frac{dv}{du}\frac{du}{dx}.$$

Since v equals e^u, dv/du equals $(e^u)' = e^u$. Since u equals $-x^2/2$, du/dx equals $-(2x)/2 = -x$. Thus, back-substituting,

$$f'(x) = \frac{dv}{dx} = (e^u)(-x) = e^{-x^2/2}(-x) = \boxed{-xe^{-x^2/2}.}$$

For the second derivative, let u and v be as defined above, and set w equals $-xv$. So w equals $f'(x)$. By the product rule,

$$\frac{dw}{dx} = (-x)'v + (-x)v' = -v - x\frac{dv}{dx}.$$

By the last paragraph,

$$\frac{dv}{dx} = -xe^{-x^2/2}.$$

Substituting in,

$$f''(x) = \frac{dw}{dx} = -e^{-x^2/2} - x(-xe^{-x^2/2}) = -e^{-x^2/2} + x^2e^{-x^2/2} = \boxed{(x^2-1)e^{-x^2/2}.}$$

For the third derivative, take u and v as above, and set z equals $(x^2 - 1)v$. So z equals $f''(x)$. By the product rule,

$$\frac{dz}{dx} = (x^2-1)'v + (x^2-1)v' = 2xv + (x^2-1)\frac{dv}{dx}.$$

By the first paragraph,

$$\frac{dv}{dx} = -xe^{-x^2/2}.$$

Substituting in,

$$f'''(x) = \frac{dz}{dx} = 2xe^{-x^2/2} + (x^2-1)(-xe^{-x^2/2}) = 2xe^{-x^2/2} + (-x^3+x)e^{-x^2/2} = \boxed{(-x^3+3x)e^{-x^2/2}.}$$

Extra credit(5 points) Only attempt this after you have completed the rest of the exam and checked your answers. For every positive integer n, show that the n^{th} derivative of $f(x)$ is of the form $f^{(n)}(x) = p_n(x)f(x)$, where $p_n(x)$ is a polynomial. Also, give a rule to compute $p_{n+1}(x)$, given $p_n(x)$.

Solution to extra credit problem The claim, proved by induction on n, is that for every positive integer n, $f^{(n)}(x)$ equals $p_n(x)$ where $p_n(x)$ is a degree n polynomial and,

$$p_{n+1}(x) = -xp_n(x) + p_n'(x).$$

The solution to Problem 2 proves this when $n = 1, 2$ and 3. Let n be a positive integer. By way of induction, assume the result is proved for n. Precisely, assume $f^{(n)}(x)$ equals $p_n(x)e^{-x^2/2}$ where $p_n(x)$ is a degree n polynomial. The goal is to prove the result for $f^{(n+1)}(x)$; precisely, $f^{(n+1)}(x)$ equals $p_{n+1}(x)e^{-x^2/2}$ for a degree $n+1$ polynomial $p_{n+1}(x)$. By definition,

$$f^{(n+1)}(x) = \frac{d}{dx}(f^{(n)}(x)).$$

By the induction hypothesis, this equals,

$$\frac{d}{dx}(p_n(x)e^{-x^2/2}).$$

Let u and v be as above, and set y equals $p_n(x)v$. So y equals $f^{(n)}(x)$. By the product rule,

$$\frac{dy}{dx} = p_n'(x)v + p_n(x)v' = p_n'(x)v + p_n\frac{dv}{dx}.$$

As computed above,

$$\frac{dv}{dx} = -xe^{-x^2/2}.$$

Substituting in,

$$\frac{dy}{dx} = p_n'(x)e^{-x^2/2} + p_n(x)(-xe^{-x^2/2}) = (-xp_n(x) + p_n'(x))e^{-x^2/2}.$$

Since $p_n(x)$ is a degree n polynomial, $p_n'(x)$ is a degree $n-1$ polynomial and $-xp_n(x)$ is a degree $n+1$ polynomial. Thus the sum $-xp_n(x) + p_n'(x)$ is a degree $n+1$ polynomial. Defining $p_{n+1}(x)$ to be,

$$p_{n+1}(x) = -xp_n(x) + p_n'(x),$$

this gives,

$$f^{(n+1)}(x) = \frac{dy}{dx} = p_{n+1}(x)e^{-x^2/2}.$$

So the result for $n+1$ follows from the result for n. Therefore the result is proved by induction on n. Moreover, this gives the inductive formula for $p_n(x)$,

$$p_{n+1}(x) = \boxed{-xp_n(x) + p_n'(x).}$$

Problem 3(15 points) A function $y = f(x)$ satisfies the implicit equation,

$$2x^3 - 9xy + 2y^3 = 0.$$

The graph contains the point $(1, 2)$. Find the equation of the tangent line to the graph of $y = f(x)$ at $(1, 2)$.

Solution to Problem 3 Differentiating both sides of the equation gives,

$$\frac{d}{dx}(2x^3 - 9xy + 2y^3) = \frac{d}{dx}(0) = 0.$$

Because the derivative is linear,

$$\frac{d}{dx}(2x^3 - 9xy + 2y^3) = 2\frac{d(x^3)}{dx} - 9\frac{d(xy)}{dx} + 2\frac{d(y^3)}{dx}.$$

Of course $d(x^3)/dx$ equals $3x^2$. By the product rule,

$$\frac{d(xy)}{dx} = \frac{d(x)}{dx}y + x\frac{dy}{dx} = y + x\frac{dy}{dx}.$$

For the last term, the chain rule gives,

$$\frac{d(y^3)}{dx} = \frac{d(y^3)}{dy}\frac{dy}{dx} = 3y^2\frac{dy}{dx}.$$

Substituting in gives,

$$\frac{d}{dx}(2x^3 - 9xy + 2y^3) = 2(3x^2) - 9\left(y + x\frac{dy}{dx}\right) + 2(3y^2)\frac{dy}{dx} = (6x^2 - 9y) + (6y^2 - 9x)\frac{dy}{dx}.$$

By the first paragraph, $d/dx(2x^3 - 9xy + 2y^3)$ equals 0. Substituting in gives the equation,

$$(6x^2 - 9y) + (6y^2 - 9x)\frac{dy}{dx} = 0.$$

Subtracting the first term from each side gives,

$$(6y^2 - 9x)\frac{dy}{dx} = (9y - 6x^2).$$

Dividing both sides by $(6y^2 - 9x)$ gives,

$$\frac{dy}{dx} = \frac{9y - 6x^2}{6y^2 - 9x} = \frac{3y - 2x^2}{2y^2 - 3x}.$$

Finally, plugging in x equals 1 and y equals 2 gives,

$$\frac{dy}{dx} = \frac{3(2) - 2(1)^2}{2(2)^2 - 3(1)} = \frac{6-2}{8-3} = \frac{4}{5}.$$

Therefore, the equation of the tangent line is,

$$y = \frac{4}{5}(x-1) + 2,$$

which simplifies to,

$$\boxed{y = (4/5)x + 6/5.}$$

Problem 4(20 points) The point $(0, 4)$ is **not** on the graph of $y = x + 1/x$, but it is contained in exactly one *tangent line* to the graph.

(a)(15 points) Find the one value of a for which the tangent line to the graph of $y = x + 1/x$ at $(a, a + 1/a)$ contains $(0, 4)$.
Hint: You do *not* need to solve a quadratic equation to find a.

Solution to (a) By the **Solution to Problem 1**, the derivative of $x + 1/x$ equals,

$$y' = 1 - \frac{1}{x^2}.$$

Thus the slope of the tangent line to the graph at $x = a$ is,

$$1 - \frac{1}{a^2} = \frac{a^2 - 1}{a^2}.$$

Therefore, the equation of the tangent line equals,

$$y = \frac{a^2-1}{a^2}(x-a) + \left(a + \frac{1}{a}\right) = \frac{a^2-1}{a^2}x + \frac{1-a^2}{a} + \left(\frac{a^2+1}{a}\right).$$

This simplifies to give the equation,

$$y = \frac{a^2-1}{a^2}x + \frac{2}{a}.$$

By hypothesis, $(0, 4)$ is contained in the tangent line. Plugging in $x = 0$ and $y = 4$ gives,

$$4 = \frac{(a^2-1)}{a^2}0 + \frac{2}{a} = \frac{2}{a}.$$

Multiplying boths sides by a gives,

$$4a = 2.$$

Dividing both sides by 4 gives,

$$a = 2/4 = \boxed{1/2.}$$

(b)(5 points) Write the equation of the corresponding tangent line.

Solution to (b) From the computation above, the equation of the tangent line at $x = a$ is,

$$y = \frac{(a^2 - 1)}{a^2}x + \frac{2}{a}.$$

Plugging in $a = 1/2$ gives,

$$a^2 - 1 = \frac{1}{4} - 1 = -\frac{3}{4},$$

$$\frac{a^2 - 1}{a^2} = \left(-\frac{3}{4}\right)(4) = -3,$$

and,

$$\frac{2}{a} = 2(2) = 4.$$

Therefore the equation of the tangent line equals,

$$\boxed{y = -3x + 4.}$$

Problem 5(25 points) In an automobile crash-test, a car is accelerated from rest at 2 m/s^2 for 5 seconds and then decelerated at $-4m/s^2$ until it strikes a barrier. The position function is,

$$s(t) = \begin{cases} t^2 & 0 \leq t < 5 \\ -2t^2 + At + B & t \geq 5 \end{cases}$$

(a)(10 points) Assuming that both $s(t)$ and $s'(t)$ are continuous at $t = 5$, determine A and B.

Solution to (a) Because $s(t)$ is continuous at $t = 5$, the left-hand limit and the right-hand limit are equal. The left-hand limit is,

$$\lim_{t \to 5^-} s(t) = \lim_{t \to 5^-} t^2 = 25.$$

The right-hand limit is,

$$\lim_{t \to 5^+} s(t) = \lim_{t \to 5^+} (-2t^2 + At + B) = -2(5)^2 + A(5) + B = -50 + 5A + B.$$

This gives the equation,

$$25 = -50 + 5A + B,$$

which simplifies to,

$$5A + B = 75.$$

The derivative $s'(t)$ equals,

$$s'(t) = \begin{cases} (t^2)' & 0 \leq t < 5 \\ (-2t^2 + At + B)' & t > 5 \end{cases}$$

which equals,

$$s'(t) = \begin{cases} 2t & 0 \le t < 5 \\ -4t + A & t > 5 \end{cases}$$

Because $s'(t)$ is continuous at $t = 5$, the left-hand limit and the right-hand limit are equal. The left-hand limit is,

$$\lim_{t \to 5^-} s'(t) = \lim_{t \to 5^-} 2t = 2(5) = 10.$$

The right-hand limit is,

$$\lim_{t \to 5^+} s'(t) = \lim_{t \to 5^+} (-4t + A) = -4(5) + A = -20 + A.$$

This gives the equation,

$$10 = -20 + A,$$

which simplfies to,

$$A = 30.$$

Plugging in $A = 30$ to the first equation gives,

$$5(30) + B = 75,$$

which simplifies to,

$$B = 75 - 5(30) = 75 - 150 = -75.$$

Therefore, the solution is,

$$\boxed{A = 30,\ B = -75.}$$

(b)(15 points) The barrier is located at $s = 33$ meters. Determine the velocity of the car when it strikes the barrier. (The quadratic polynomial has whole number roots.)

Solution to (b) For $t > 5$, the equation for displacement is,

$$s(t) = -2t^2 + 30t - 75.$$

The moment T when the car strikes the barrier is the solution of the equation $s(T) = 33$,

$$-2T^2 + 30T - 75 = 33.$$

Subtracting 33 from each side gives the equation,

$$-2T^2 + 30T - 108 = 0.$$

Dividing each side by -2 gives the equation,

$$T^2 - 15T + 54 = 0.$$

The fraction 54 factors as 2×27, 3×18 and 6×9. In the last case, the sum of the factors is $+15$. Thus the quadratic polynomial factors as,

$$T^2 - 15T + 54 = (T - 6)(T - 9).$$

The two possible solutions of $(T - 6)(T - 9) = 0$ are $T = 6$ and $T = 9$. Since the car cannot crash twice, the car crashes at the moment,

$$T = 6.$$

For $t > 5$, the equation of $v(t) = s'(t)$ was calculated above to be,

$$s'(t) = -4t + A = -4t + 30.$$

Plugging in $t = T = 6$ gives,

$$s'(6) = -4(6) + 30.$$

Therefore, at the moment the car crashes into the barrier, the velocity is,

$$\boxed{6 \text{ meters/second.}}$$

Problem 6(15 points) For each of the following functions, compute the derivative. Show all work.

(a)(4 points) $y = (e^x - e^{-x})/(e^x + e^{-x})$

Solution to (a) Set $u = e^x - e^{-x}$ and $v = e^x + e^{-x}$. Then $y = u/v$. By the quotient rule, the derivative is,

$$\frac{dy}{dx} = \frac{1}{v^2}(\frac{du}{dx}v - u\frac{dv}{dx}).$$

Using the chain rule,

$$\frac{du}{dx} = e^x(1) - e^{-x}(-1) = e^x + e^{-x} = v.$$

Similarly,

$$\frac{dv}{dx} = e^x(1) + e^{-x}(-1) = e^x - e^{-x} = u.$$

Plugging in gives,

$$\frac{dy}{dx} = \frac{1}{v^2}(v^2 - u^2).$$

Expanding gives,

$$v^2 - u^2 = (e^x - e^{-x})^2 - (e^x + e^{-x})^2 = [(e^x)^2 - 2e^x e^{-x} + (e^{-x})^2] - [(e^x)^2 + 2e^x e^{-x} + (e^{-x})^2].$$

Cancelling, this gives,

$$v^2 - u^2 = -4e^x e^{-x} = -4.$$

Therefore, the derivative equals,

$$\frac{dy}{dx} = -4/v^2 = \boxed{-4/(e^x - e^{-x})^2.}$$

(b)(3 points) $y = x\ln(x) - x$

Solution to (b) Because the derivative is linear,

$$y' = (x\ln(x))' - (x)' = (x\ln(x))' - 1.$$

By the product rule,

$$(x\ln(x))' = (x)'\ln(x) + x(\ln(x))' = 1\ln(x) + x\frac{1}{x} = \ln(x) + 1.$$

Therefore the derivative is $\ln(x) + 1 - 1$, which is,

$$\boxed{y' = \ln(x).}$$

(c)(3 points) $y = \sqrt{1 + x^{1234}}$

Solution to (c) Set u equals x^{1234}. Set v equals $1 + u$, which equals $1 + x^{1234}$. Then y equals $v^{1/2}$, which equals $(1 + x^{1234})^{1/2}$. By the chain rule,

$$\frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{du}\frac{du}{dx}.$$

By the formula for the derivative of x^a,

$$\frac{du}{dx} = 1234x^{1233}, \frac{dv}{du} = 1, \frac{dy}{dv} = \frac{1}{2}v^{-1/2}.$$

Thus the chain rule gives,

$$\frac{dy}{dx} = \frac{1}{2}v^{-1/2}(1)(1234x^{1233}) = \frac{1}{2}(1 + x^{1234})^{-1/2}(1234x^{1233}).$$

This simplifes to give,

$$\boxed{y' = 617x^{1233}/\sqrt{1 + x^{1234}}.}$$

(d)(5 points) $y = \log_{10}(x^3 + 3x)$.

Solution to (d) The inner term factors as $x^3 + 3x = x(x^2 + 3)$. Since $\log_{10}(AB)$ equals $\log_{10}(A) + \log_{10}(B)$, the expression for y simplifies to,

$$y = \log_{10}(x(x^2 + 3)) = \log_{10}(x) + \log_{10}(x^2 + 3).$$

Because the derivative is linear,

$$y' = (\log_{10}(x))' + (\log_{10}(x^2+3))'.$$

The formula for the derivative of a logarithm function is,

$$\frac{d(\log_a(x))}{dx} = \frac{1}{\ln(a)x}.$$

Thus,

$$(\log_{10}(x))' = \frac{1}{\ln(10)x}.$$

For the second term, set u equals x^2+3. And set v equals $\log_{10}(u) = \log_{10}(x^2+3)$. By the chain rule,

$$\frac{d}{dx}(\log_{10}(x^2+3)) = \frac{dv}{dx} = \frac{dv}{du}\frac{du}{dx}.$$

By the formula for the derivative of a logarithm function,

$$\frac{dv}{du} = \frac{d}{du}(\log_{10}(u)) = \frac{1}{\ln(10)u}.$$

And, of course,

$$\frac{du}{dx} = (x^2+3)' = 2x.$$

Thus, the derivative is,

$$\frac{dv}{dx} = \frac{1}{\ln(10)u}(2x) = \frac{1}{\ln(10)(x^2+3)}(2x).$$

Putting the pieces together,

$$y' = \frac{1}{\ln(10)x} + \frac{2x}{\ln(10)(x^2+3)}.$$

This simplifies to give,

$$\boxed{y' = 3(x^2+1)/(\ln(10)x(x^2+3)).}$$