18.01 Calculus | **Jason Starr**
Due by 2:00pm sharp | **Fall 2005**
Friday, Dec. 9, 2005

Solutions to Problem Set 8

Late homework policy. Late work will be accepted only with a medical note or for another Institute-approved reason.

Cooperation policy. You are encouraged to work with others, but the final write-up must be entirely your own and based on your own understanding. You may not copy another student's solutions. And you should not refer to notes from a study group while writing up your solutions (if you need to refer to notes from a study group, it isn't really "your own understanding").

Part I. These problems are mostly from the textbook and reinforce the basic techniques. Occasionally the solution to a problem will be in the back of the textbook. In that case, you should work the problem first and only use the solution to check your answer.

Part II. These problems are not taken from the textbook. They are more difficult and are worth more points. When you are asked to "show" some fact, you are not expected to write a "rigorous solution" in the mathematician's sense, nor a "textbook solution". However, you should write a clear argument, using English words and complete sentences, that would convince a typical Calculus student. (Run your argument by a classmate; this is a good way to see if your argument is reasonable.) Also, for the grader's sake, try to keep your answers as short as possible (but don't leave out *important* steps).

Part I(20 points)

- **(a)** (4 points) p. 403, Section 12.2, Problem 14
- **(b)** (4 points) p. 408, Section 12.3, Problem 38
- **(c)** (4 points) p. 413, Section 12.4, Problem 8
- **(d)** (4 points) p. 413, Section 12.4, Problem 13
- **(e)** (4 points) p. 414, Section 12.4, Problem 20

Solution to (a) Since $\ln(\tan(\pi/4)) = 0$, and $\sin(\pi/4) - \cos(\pi/4) = 0$, this limit has the "$\frac{0}{0}$" indeterminate form, which justifies our use of the L'hospital's rule:

$$\begin{aligned}
\lim_{x\to\pi/4} \frac{\ln(tanx)}{\sin x - \cos x} &= \lim_{x\to\pi/4} \frac{\frac{d}{dx}\ln(tanx)}{\frac{d}{dx}(\sin x - \cos x)} = \lim_{x\to\pi/4} \frac{\frac{\sec^2 x}{\tan x}}{\cos x + \sin x} \\
&= \lim_{x\to\pi/4} \frac{\frac{1}{\sin x \cos x}}{\cos x + \sin x} = \frac{\frac{1}{\sin(\pi/4)\cos(\pi/4)}}{\cos \pi/4 + \sin \pi/4} = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\frac{1}{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}} \\
&= \frac{1}{2}\frac{\sqrt{2}}{2} = \boxed{\frac{\sqrt{2}}{4}}
\end{aligned} \tag{1}$$

Solution to (b) By plugging in $x = \pi/2$, we see that the limit becomes "1^∞", which IS an indeterminate form. In order to convert this form to one of the indeterminate forms "$\frac{0}{0}$", or "$\frac{\infty}{\infty}$", for which we can use the L'hospital's rule, let us write

$$\begin{aligned}\lim_{x\to\pi/2}(\sin x)^{\tan x} &= \lim_{x\to\pi/2} e^{\ln(\sin x)^{\tan x}} = \lim_{x\to\pi/2} e^{\tan x \ln(\sin x)} \\ &= e^{\lim_{x\to\pi/2}\tan x \ln(\sin x)}\end{aligned} \tag{2}$$

Therefore we are interested in the limit

$$\lim_{x\to\pi/2} \tan x \ln(\sin x) = \lim_{x\to\pi/2} \frac{\ln(\sin x)}{\cot x}$$

which, when written in this last form, has the "$\frac{\infty}{\infty}$" indeterminancy, therefore we can invoke L'Hospital's rule to yield

$$\lim_{x\to\pi/2}\frac{\ln(\sin x)}{\cot x} = \lim_{x\to\pi/2}\frac{\frac{d}{dx}\ln(\sin x)}{\frac{d}{dx}\cot x} = \lim_{x\to\pi/2}\frac{\frac{\cos x}{\sin x}}{-\csc^2 x} = \lim_{x\to\pi/2} -\cos x \sin x = -(0)(1) = 0$$

Therefore we have

$$\lim_{x\to\pi/2}(\sin x)^{\tan x} = e^{\lim_{x\to\pi/2}\tan x\ln(\sin x)} = e^0 = \boxed{1}.$$

Solution to (c) We already know that

$$\int e^{-x}\cos x dx = \frac{1}{2}(\sin x - \cos x) + C \tag{3}$$

In case you forgot this formula, its derivation is as follows: Let $u = e^{-x}$ and $dv = \cos x dx$, and integrate by parts to obtain

$$\int e^{-x}\cos x dx = e^{-x}\sin x - \int -e^{-x}\sin x dx$$

One more integration parts, now with $u = e^{-x}$, and $dv = \sin x dx$ yields

$$\int e^{-x}\cos x dx = e^{-x}\sin x - e^{-x}\cos x - \int e^{-x}\cos x dx$$

which, upon rearranging and dividing both sides by 2, yields the formula (3). Therefore

$$\begin{aligned}\int_0^\infty e^{-x}\cos x dx &= \lim_{b\to\infty}\int_0^b e^{-x}\cos x dx \\ &= \lim_{b\to\infty}\frac{1}{2}(\sin x - \cos x)|_0^b = \frac{1}{2} + \lim_{b\to\infty}\frac{1}{2}e^{-b}(\sin b - \cos b) \\ &= \boxed{\tfrac{1}{2}}\end{aligned} \tag{4}$$

Note that the limit $\lim_{b\to\infty} e^{-b}(\sin b - \cos b) = 0$, because e^{-b} vanishes as b tends to ∞, and $(\sin b - \cos b)$ remains bounded, namely its values are always in $[-2, 2]$ for all values of b. More precisely, this is a consequence of the squeezing lemma, and

$$-2e^{-b} \leq e^{-b}(\sin b - \cos b) \leq 2e^{-b}$$

with the observation that both $\pm 2e^{-b}$ tend to zero as b tends to ∞.

Solution to (d) First off, we observe that the required integral is improper, because $\ln x/\sqrt{x}$ blows up (or down?) to $-\infty$ at $x = 0^+$. Integrating by parts with $u = \ln x$ and $dv = dx/\sqrt{x}$, we obtain

$$\int \frac{\ln x}{\sqrt{x}} dx = 2\sqrt{x} \ln x dx - \int \frac{2}{\sqrt{x}} dx = 2\sqrt{x}(\ln x - 2) + C$$

Therefore it is straightforward to evaluate the required improper integral as

$$\begin{aligned} \int_0^2 \frac{\ln x}{\sqrt{x}} dx &= \lim_{u\to 0^+} \int_u^2 \frac{\ln x}{\sqrt{x}} dx = \lim_{u\to 0^+} (2\sqrt{x}(\ln x - 2))|_u^2 \\ &= 2\sqrt{2}(\ln 2 - 2) + 4 \lim_{u\to 0^+} u^{1/2} - 2 \lim_{u\to 0^+} \frac{\ln u}{u^{-1/2}} \end{aligned} \tag{5}$$

While the first limit is clearly zero, we need to resort to our friend L'hospital for the evaluation of the second one, as it is in the indeterminate form "$\frac{\infty}{\infty}$". This yields

$$\lim_{u\to 0^+} \frac{\ln u}{u^{-1/2}} = \lim_{u\to 0^+} \frac{\frac{d}{du} \ln u}{\frac{d}{du} u^{-1/2}} = \lim_{u\to 0^+} \frac{\frac{1}{u}}{\frac{-1}{2} u^{-3/2}} = \lim_{u\to 0^+} -2u^{1/2} = 0$$

Therefore, the improper integral converges (since we did not get any divergent limits), and it is equal to

$$\int_0^2 \frac{\ln x}{\sqrt{x}} dx = \boxed{2\sqrt{2}(\ln 2 - 2)}$$

Solution to (e) Recall the disk method: we have $dV = \pi y^2 dx$. Now that we have infinite domain, we need to be a little more careful:

$$\begin{aligned} \int_1^\infty \pi y^2 dx &= \lim_{b\to\infty} \int_1^b \pi y^2 dx \\ &= \lim_{b\to\infty} \int_1^b \frac{16\pi}{9} x^{-3/2} dx = \lim_{b\to\infty} (-\frac{32\pi}{9} x^{-1/2})|_1^b \\ &= \lim_{b\to\infty} \frac{32\pi}{9}(1 - \frac{1}{\sqrt{b}}) = \boxed{\frac{32\pi}{9}} \end{aligned} \tag{6}$$

Part II(30 points)

Problem 1(15 points) Solve Problem 21, p. 414, §12.4 of the textbook. First solve **(e)** above.

Solution to Problem 1 We recall that the infinitesimal element for the surface area of revolution is $dA = 2\pi y ds$, where $ds = \sqrt{1 + (\frac{dy}{dx})^2}dx$. Therefore $dA = 2\pi\frac{4}{3}x^{-3/4}\sqrt{1+x^{-7/2}}$, and hence the surface area can be written as

$$\begin{aligned} A = \int_1^\infty dA &= \lim_{b\to\infty}\int_1^b \frac{8\pi}{3}x^{-3/4}\sqrt{1+x^{-7/2}}dx \\ &\geq \lim_{b\to\infty}\int_1^b \frac{8\pi}{3}x^{-3/4}dx = \lim_{b\to\infty}\frac{32\pi}{3}x^{1/4}|_1^b = \frac{32\pi}{3}\lim_{b\to\infty}(b^{1/4}-1) \end{aligned} \tag{7}$$

which diverges to infinity. By the comparison test, we conclude that the integral $\int_1^\infty dA$ diverges, too.

Problem 2(15 points) Solve Problem 2 from Exam 4 using the disk method. In other words, use the disk method to compute the volume of the solid obtained by revolving about the y-axis the region in the first quadrant bounded by the x-axis, the line $x = 0$, the line $x = a$ and the curve,

$$y = \frac{ab}{x} - b.$$

Solution to Problem 2 Since we are revolving around the y-axis, $dV = \pi x^2 dy = \pi\frac{a^2b^2}{(y+b)^2}dy$. We also observe that, when $x = a$, we have $y = 0$ and as $x \to 0$, we have $y \to \infty$. Therefore the limits of integration are 0 and ∞. Hence the volume can be calculated as

$$\begin{aligned} V = \int_0^\infty dA &= \lim_{t\to\infty}\int_0^t dV = \lim_{t\to\infty}\int_0^t \frac{\pi a^2b^2}{(y+b)^2}dy = \lim_{t\to\infty}(-\frac{\pi a^2b^2}{y+b})|_t^0 \\ &= \lim_{t\to\infty}\pi a^2b^2(\frac{1}{b}-\frac{1}{b+t}) = \frac{\pi a^2b^2}{b} = \boxed{\pi a^2 b} \end{aligned} \tag{8}$$