

Solutions to Problem Set 6

Late homework policy. Late work will be accepted only with a medical note or for another Institute-approved reason.

Cooperation policy. You are encouraged to work with others, but the final write-up must be entirely your own and based on your own understanding. You may not copy another student's solutions. And you should not refer to notes from a study group while writing up your solutions (if you need to refer to notes from a study group, it isn't really "your own understanding").

Part I. These problems are mostly from the textbook and reinforce the basic techniques. Occasionally the solution to a problem will be in the back of the textbook. In that case, you should work the problem first and only use the solution to check your answer.

Part II. These problems are not taken from the textbook. They are more difficult and are worth more points. When you are asked to "show" some fact, you are not expected to write a "rigorous solution" in the mathematician's sense, nor a "textbook solution". However, you should write a clear argument, using English words and complete sentences, that would convince a typical Calculus student. (Run your argument by a classmate; this is a good way to see if your argument is reasonable.) Also, for the grader's sake, try to keep your answers as short as possible (but don't leave out *important* steps).

Part I(20 points)

- (a) (2 points) p. 235, Section 7.4, Problem 18(a) Ignore the introductory
- (b) (2 points) p. 235, Section 7.4, Problem 18(b) physics note/"problem"
- (c) (2 points) p. 591, Section 17.1, Problem 12(a)
- (d) (2 points) p. 591, Section 17.1, Problem 12(b)
- (e) (2 points) p. 240, Section 7.5, Problem 10
- (f) (2 points) p. 244, Section 7.6, Problem 7
- (g) (2 points) p. 563, Section 16.1, Problem 7
- (h) (2 points) p. 567, Section 16.2, Problem 1 12 and 15 are also fun
- (i) (2 points) p. 579, Section 16.4, Problem 14
- (j) (2 points) p. 582, Section 16.5, Problem 7

Solution to (a) For the given profile $y = \frac{\omega^2}{2g}x^2 + h$, we can calculate the volume of the water by the shell method: We have

$$dV = 2\pi xy dx = 2\pi\left(\frac{\omega^2}{2g}x^3 + hx\right)dx$$

therefore, the volume is

$$V_0 = \int dV = \int_0^r 2\pi\left(\frac{\omega^2}{2g}x^3 + hx\right)dx = 2\pi\left(\frac{\omega^2 x^4}{8g} + \frac{hx^2}{2}\right)\Big|_0^r = 2\pi\left(\frac{\omega^2 r^4}{8g} + \frac{hr^2}{2}\right)$$

We are required to express h in terms of ω , which is easy to read from the last equation:

$$h = \frac{V_0}{\pi r^2} - \frac{\omega^2 r^2}{4g}$$

Note that $h(\omega) = h(0) - \frac{V_0}{\pi r^2}$, therefore

$$\begin{aligned} y_{min}(\omega) &= y(\omega)|_{x=0} = h(\omega) = h(0) - \frac{\omega^2 r^2}{4g} \\ y_{max}(\omega) &= \frac{\omega^2 r^2}{2g} + h(\omega) = \left(h(0) - \frac{\omega^2 r^2}{4g}\right) + \frac{\omega^2 r^2}{2g} = h(0) + \frac{\omega^2 r^2}{4g} \end{aligned}$$

Solution to (b) We are given that the can is half filled, which means $h(\omega = 0) = \frac{1}{2}H$. The water will begin spilling when

$$y_{max}(\omega) = h(0) + \frac{\omega^2 r^2}{4g} = \frac{1}{2}H + \frac{\omega^2 r^2}{4g} = H$$

which happens when $\frac{\omega^2 r^2}{4g} = \frac{1}{2}H$. The bottom will start getting exposed when

$$y_{min}(\omega) = h(0) - \frac{\omega^2 r^2}{4g} = \frac{1}{2}H - \frac{\omega^2 r^2}{4g} = 0$$

which happens again when $\frac{\omega^2 r^2}{4g} = \frac{1}{2}H$. Hence, both the water starts spilling and the bottom gets exposed exactly for the same ω

Solution to (c) From $t = \frac{y}{1+x}$, we obtain $y = t(1+x)$, which, together with $x^2 + y^2 = 1$ yields $x^2 + t^2(1+x)^2 - 1 = 0$, which can be written as

$$(1+t^2)x^2 + 2t^2x + (t^2-1) = (1+t^2)\left(x^2 + 2\frac{t^2}{1+t^2}x + \frac{t^2-1}{1+t^2}\right) = 0$$

which, after completing to perfect square, translates into

$$\left(x + \frac{t^2}{1+t^2}\right)^2 - \frac{1}{t^2+1} = 0$$

The solution to this equation is easily seen to be $x = -\frac{t^2}{1+t^2} \pm \frac{1}{t^2+1}$, in other words $x = \frac{1-t^2}{1+t^2}$ or $x = -1$. Substituting these values of x into $y = t(1+x)$, we obtain $y = \frac{2t}{1+t^2}$ or $y = 0$ when $x = -1$. Therefore, if $x \neq -1$, then

$$x = \frac{1-t^2}{1+t^2} \quad y = \frac{2t}{1+t^2}$$

for $t = \frac{y}{1+x}$.

Solution to (d) If x and y are fractions, so is $t = \frac{y}{1+x}$, and if t is a fraction, then so are $x = \frac{1-t^2}{1+t^2}$ and $y = \frac{2t}{1+t^2}$.

Solution to (e) The curve $y = (a^{2/3} - x^{2/3})^{3/2}$ constitutes one quarter of the astroid, and can be sketched relatively easily. The figure of the full astroid is given on the course webpage.

From $y = (a^{2/3} - x^{2/3})^{3/2}$, we get $y' = \frac{3}{2}(a^{2/3} - x^{2/3})^{1/2}(\frac{2}{3}x^{-1/3})$. Therefore

$$ds = \sqrt{1 + (y')^2} dx = \sqrt{\frac{x^{2/3}}{x^{2/3}} + \frac{a^{2/3} - x^{2/3}}{x^{2/3}}} dx = \frac{a^{1/3}}{x^{1/3}} dx$$

Therefore, one quarter of the length is

$$\int_0^a a^{1/3} x^{-1/3} dx = \frac{3}{2} a^{1/3} x^{2/3} \Big|_0^a = \frac{3}{2} a$$

Hence the total length is **6a**. Alternative, one can use the parametrization

$$x = a \cos^3 t \quad y = a \sin^3 t$$

for the astroid. Here t ranges in $[0, 2\pi]$. We then calculate $ds = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt = |\cos t \sin t| dt$ which can be easily integrated from 0 to $\pi/2$ and multiplied by 4 to yield the same result.

Solution to (f) We can rewrite $x^2 = 4py$ as $y = \frac{1}{4p}x^2$, and differentiating with respect to x , we obtain $y' = \frac{1}{2p}x$, and therefore $s = \sqrt{1 + (y')^2} dx = \sqrt{1 + (\frac{x}{2p})^2} dx$. Hence the surface area is

$$S = \int 2\pi x ds = \int_{x=0}^{x=2p} 2\pi x \sqrt{1 + (\frac{x}{2p})^2} dx$$

In order to compute this integral, we introduce the variable $u = 1 + (\frac{x}{2p})^2$, which has differential $du = \frac{x}{p^2} dx$, and change the limits of integration to $u = 1$ and $u = 2$. Then, we can write the surface area as

$$S = \int_1^2 2\pi u^{1/2} (2p^2 du) = 4\pi p^2 (\frac{2}{3} u^{3/2} \Big|_1^2) = \frac{8\pi(2\sqrt{2})-1}{3} p^2$$

Alternatively, we can switch the roles of x and y and use the formula for the surface area. We express the curve as $x = (4py)^{1/2}$, which yields $\frac{dx}{dy} = \frac{1}{2}(4py)^{-1/2} 4p$. We then calculate $ds = \sqrt{1 + (\frac{dx}{dy})^2} dy = \sqrt{1 + \frac{p}{y}} dy$. The surface area is therefore given by

$$\begin{aligned} S &= \int 2\pi x ds = \int_{y=0}^{y=p} 2\pi \sqrt{4py} \sqrt{1 + \frac{p}{y}} dy = 2\pi \cdot 2\sqrt{p} \int_0^p \sqrt{y+p} dy \\ &= 4\pi \sqrt{p} (\frac{2}{3} (y+p)^{3/2} \Big|_0^p) = \frac{8\pi}{3} \sqrt{p} ((2p)^{3/2} - p^{3/2}) = \frac{8\pi(2\sqrt{2})-1}{3} p^2 \end{aligned}$$

Solution to (g) By multiplying both sides of the equation $r = 4(\cos \theta + \sin \theta)$ by r , we obtain $r^2 = 4(r \cos \theta + r \sin \theta)$, which can be written in cartesian form as

$$x^2 + y^2 = 4(x + y)$$

This is the same as $x^2 - 4x + 4 + y^2 - 4y + 4 = 8$, which is,

$$(x - 2)^2 + (y - 2)^2 = 8$$

This is a circle of radius $2\sqrt{2}$ centered at $(2, 2)$. A sketch can be found on the course webpage.

Solution to (h) The sketches can be found on the course webpage.

Solution to (i) Differentiating, we obtain $r' = -2a \sin \theta$. Therefore

$$ds = \sqrt{(r')^2 + r^2} d\theta = \sqrt{4a^2 \cos^2 \theta + 4a^2 \sin^2 \theta} d\theta = 2ad\theta$$

Thus the length of the circle is given by

$$\int_{\theta=-\pi/2}^{\theta=\pi/2} 2ad\theta = 2a\pi$$

Solution to (j) From a sketch of the curve, it is easy to see that negative values for r correspond to the inner curve, whereas positive values for r correspond to the outer curve. Therefore the inner curve is given by the part $-\frac{2\pi}{3} \leq \theta \leq \frac{2\pi}{3}$, and the outer curve is given by the part $\frac{2\pi}{3} \leq \theta \leq \frac{4\pi}{3}$. Remembering that $dA = \frac{1}{2}r^2 d\theta$, the outer area is given by

$$\begin{aligned} O &= \int_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}} \frac{(1 + 2 \cos \theta)}{2} d\theta = \frac{1}{2} \int_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}} (1 + 4 \cos \theta + 4 \cos^2 \theta) d\theta \\ &= \frac{1}{2} \int_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}} (1 + 4 \cos \theta + 2(1 + \cos 2\theta)) d\theta = \frac{1}{2} (3\theta + 4 \sin \theta + \sin 2\theta) \Big|_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}} = 2\pi + \frac{3\sqrt{3}}{2} \end{aligned}$$

Similarly, the inner area can be written as

$$I = \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{(1 + 2 \cos \theta)}{2} d\theta = \pi - \frac{3\sqrt{3}}{2}$$

Therefore the net area, which is the difference between the outer and the inner area is just $\pi + 3\sqrt{3}$.

Part II(30 points)

Problem 1(10 points) Solve Problem 41 on p. 256. In other words, compute the volume of a torus using the shell method.

Solution to Problem 1 We calculate the volume in two parts: the volume obtained by revolution of the *inner* part where $x < b$ and volume obtained by revolution of the *outer* part where $x > b$. As shown in the figure on the course webpage, the height of a slice is $2y = 2\sqrt{a^2 - z^2}$. Let's call the inner and outer x -coordinates of the slices x_i and x_o . Then $x_i = b - z$ and $x_o = b + z$ where z is the distance from the line $x = b$. Moreover, $dV_{inner} = 2\pi x_i y dz$, and $dV_{outer} = 2\pi x_o y dz$, hence the total volume of the two slices is

$$dV = dV_i + dV_o = 2\pi[(b - z) + (b + z)]2\sqrt{a^2 - z^2} dz = 2\pi \cdot 2b \cdot 2\sqrt{a^2 - z^2} dz$$

Since \int_0^a being one quarter of the area of the circle with radius a , is equal to $\int_0^a \sqrt{a^2 - z^2} dz = \frac{1}{4}\pi a^2$, we obtain

$$V = 8\pi b \frac{\pi a^2}{4} = (2\pi b)(\pi a^2)$$

Problem 2 (20 points) The *three-leaved rose* is the polar curve,

$$r = \cos(3\theta).$$

(a) (5 points) Sketch the polar curve for $-\pi/2 \leq \theta \leq \pi/2$. In a separate box, list the angles where r is maximum, the angles where r is 0, and the slope of the curve at each point where r is 0.

Solution to (a) We first note that since $-\pi/2 \leq \theta \leq \pi/2$, the argument of the cosine, which is 3θ , ranges in the interval $[-\frac{3\pi}{2}, \frac{3\pi}{2}]$. In this interval, cosine takes its maximum value, which is 1, at $3\theta = 0$, it also takes its minimum, i.e. the value -1 twice, at points $3\theta = \pm\pi$. Since cosine vanishes at $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$, solving $3\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$, we obtain the zeros of r , which are $\pm\frac{\pi}{6}, \pm\frac{\pi}{2}$. We can now make a table examining the sign of r .

θ	$-\frac{\pi}{2}$			$-\frac{\pi}{6}$			$\frac{\pi}{6}$			$\frac{\pi}{2}$
r	0	-	-	0	+	+	0	-	-	0

The sketch of the 3-leaved rose can be found on the course webpage.

(b) (5 points) Compute the area of the leaf of the rose that contains the point $(\theta = 0, r = 1)$.

Solution to (b) Recalling that $dA = \frac{1}{2}r^2 d\theta$, and the double angle formula $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ we can compute the expression for the area as

$$\begin{aligned} A &= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2} \cos^2 3\theta d\theta = 2 \int_0^{\frac{\pi}{6}} \frac{1 + \cos 6\theta}{4} d\theta \\ &= 2 \left(\frac{\theta}{4} + \frac{\sin 6\theta}{24} \right) \Big|_0^{\frac{\pi}{6}} = 2 \frac{\pi}{24} = \frac{\pi}{12} \end{aligned}$$

(c) (5 points) Read Problem 11, §9.1, p. 300. You may use the solution from the back of the book; the problem is very similar to Problem 12, which was Part I, (h) from **Problem Set 2**. Using the

solution, find an implicit equation for the Cartesian form of the curve, e.g. some polynomial in x and y equals 0.

Solution to (c) We first need to express $r = \cos 3\theta$ in terms of $\cos \theta$ and $\sin \theta$. The formula $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$ can be found in many calculus books. We therefore have the equation $r = 4 \cos^3 \theta - 3 \cos \theta$. We proceed to multiply by r^3 to obtain

$$r^4 = 4(r \cos \theta)^3 - 3r^2(r \cos \theta)$$

Substituting $x = r \cos \theta$ and $r^2 = x^2 + y^2$, we obtain

$$(x^2 + y^2)^2 = 4x^3 - 3(x^2 + y^2)x$$

(d)(5 points) Using your solution to (c), write down an integral of the form,

$$2 \int_{x=0}^{x=1} f(x) dx,$$

for the area of the leaf in (b). Do not attempt to solve this integral; you already computed its value.

Solution to (d) Solving the expression found in (c) for y yields

$$y = \left[\frac{3x}{2} \left(1 + \frac{16x}{9} \right)^{1/2} - \frac{3x}{2} - x^2 \right]^{1/2}$$

Therefore, another expression for the area found in (b) would be

$$\int_{x=0}^{x=1} \left[\frac{3x}{2} \left(1 + \frac{16x}{9} \right)^{1/2} - \frac{3x}{2} - x^2 \right]^{1/2} dx$$