18.01 Calculus
Due by 2:00pm sharp
Friday, Sept. 30, 2005

Jason Starr
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Solutions to Problem Set 2

Part I/Part II

Part I(20 points)

(a)	(2 points)	p. 97,	Section 3.3,	Problem 44
(b)	(2 points)	p. 107,	Section 3.5,	Problem 15
(c)	(2 points)	p. 110,	Section 3.6,	Problem 3(c)
(d)	(2 points)	p. 263,	Section 8.2,	Problem 6
(e)	(2 points)	p. 264,	Section 8.2,	Problem 11
(f)	(2 points)	p. 269,	Section 8.3,	Problem 4
(g)	(2 points)	p. 277,	Section 8.4,	Problem 19(b)
(h)	(2 points)	p. 300,	Section 9.1,	Problem 12
(i)	(2 points)	p. 305,	Section 9.2,	Problem 16
(j)	(2 points)	p. 311,	Section 9.4,	Problem 8

Solution (a) Let u equal $8 - x^2$, and let $v = u^5$. Then y equals,

$$y = \frac{x}{v}.$$

By the quotient rule,

$$y' = \frac{1}{v^2}((x)'v - x(v')) = \frac{1}{v^2}(v - x(v')).$$

By the chain rule,

$$v' = \frac{dv}{dx} = \frac{dv}{du} \cdot \frac{du}{dx} = (5u^4)(-2x).$$

When x equals 3, u equals $8-(3)^2 = -1$ and v equals $(-1)^5 = -1$. Thus $v'(3)$ equals $(5(-1)^4)(-2 \cdot 3) = -30$. Thus, $y'(3)$ equals,

$$y'(3) = \frac{1}{(-1)^2}((-1) - (3)(-30)) = -1 + 90 = 89.$$

Therefore, the slope of the tangent line at $(3, -3)$ is,

$$y = 89(x-3) + (-3), \quad \boxed{y = 89x - 270}.$$

Solution (b) Implicit differentiation gives,

$$\frac{d}{dx}\left(\frac{1-y}{1+y}\right) = \frac{d(x)}{dx} = 1.$$

By the chain rule,

$$\frac{d}{dx}\left(\frac{1-y}{1+y}\right) = \frac{d}{dy}\left(\frac{1-y}{1+y}\right)\frac{dy}{dx}.$$

By the quotient rule,

$$\frac{d}{dy}\left(\frac{1-y}{1+y}\right) = \frac{1}{(1+y)^2}((1-y)'(1+y)-(1-y)(1+y)') = \frac{1}{(1+y)^2}((-1)(1+y)-(1-y)(1)) = \frac{-2}{(1+y)^2}.$$

Thus, implicit differentiation gives,

$$\frac{-2}{(1+y)^2}\frac{dy}{dx} = 1,$$

or,

$$\frac{dy}{dx} = \boxed{-(1+y)^2/2}.$$

To solve for x, multiply both sides of the equation by $1+y$ to get,

$$1-y = x(1+y) = x+xy.$$

Add $y-x$ to each side of the equation to get,

$$1-x = xy+y = (x+1)y.$$

Divide each side of the equation to get,

$$y = \frac{1-x}{1+x}.$$

By the quotient rule,

$$y' = \frac{1}{(1+x)^2}((1-x)'(1+x)-(1-x)(1+x)') = \frac{1}{(1+x)^2}((-1)(1+x)-(1-x)(1)) = \boxed{-2/(1+x)^2}.$$

Since $y = (1-x)/(1+x)$, $1+y$ equals,

$$1+y = \frac{1+x}{1+x} + \frac{1-x}{1+x} = \frac{(1+x)+(1-x)}{1+x} = \frac{2}{1+x}.$$

Thus,

$$\frac{-1}{2}(1+y)^2 = \frac{-1}{2}\left(\frac{2}{1+x}\right)^2 = \frac{-1}{2}\frac{4}{(1+x)^2}.$$

Therefore,

$$\frac{-1}{2}(1+y)^2 = \frac{-2}{(1+x)^2}.$$

So the two answers for y' are equivalent.

Solution (c) The fraction simplifies,

$$y = \frac{x}{1+x} = \frac{(1+x)-1}{1+x} = \frac{1+x}{1+x} - \frac{1}{1+x} = 1 - \frac{1}{1+x}.$$

Let z equal $1/(1+x)$. Then y equals $1-z$. So y' equals $-z'$. So y'' equals $-z''$. Clearly, for every positive integer n, $y^{(n)}$ equals $-z^{(n)}$. By the same argument as in Example 2 on p. 109,

$$z^{(n)} = (-1)^n n!(x+1)^{-(n+1)}.$$

Thus, for every positive integer n,

$$y^{(n)} = \boxed{(-1)^{n+1} n!(x+1)^{-(n+1)}}.$$

Solution (d) The equation,

$$y = \log_a(x + \sqrt{x^2-1}),$$

is equivalent to the equation,

$$a^y = x + \sqrt{x^2-1}.$$

Subtract x from each side to get,

$$a^y - x = \sqrt{x^2-1},$$

and then square each side to get,

$$(a^y - x)^2 = x^2 - 1.$$

Expanding the left-hand-side gives,

$$(a^y)^2 - 2xa^y + x^2 = x^2 - 1.$$

Cancelling x^2 from each side gives,

$$(a^y)^2 - 2xa^y = -1.$$

Adding $2xa^y + 1$ to each side of the equation gives,

$$(a^y)^2 + 1 = 2xa^y.$$

The expression $2a^y$ is always nonzero. Thus it is valid to divide each side by $2a^y$, giving,

$$x = [(a^y)^2 + 1]/(2a^y) = [a^y + 1/a^y]/2.$$

Of course $1/a^y$ equals a^{-y}. So this simplifies to,

$$x = \boxed{(a^y + a^{-y})/2}.$$

Solution (e) Because $1/2$ is less than 1, $\log(1/2)$ is less than $\log(1) = 0$. Thus $\log(1/2)$ is negative. For every pair of real numbers $a < b$ and every negative number c, ac is **greater than** bc, not less than bc. Therefore, the correct inequality is,

$$1 \cdot \log \frac{1}{2} > 2 \cdot \log \frac{1}{2}.$$

The remainder of the argument is correct, and eventually leads to the true inequality,

$$\frac{1}{2} > \frac{1}{4}.$$

Solution (f) By the product rule,

$$\frac{d}{dx}(x^2 e^{-x^2}) = \frac{d(x^2)}{dx} e^{-x^2} + x^2 \frac{d(e^{-x^2})}{dx} = 2xe^{-x^2} + x^2 \frac{d(e^{-x^2})}{dx}.$$

Let u equal $-x^2$ and let v equal e^u. Thus v equals e^{-x^2}. By the chain rule,

$$\frac{dv}{dx} = \frac{dv}{du}\frac{du}{dx}.$$

Also,

$$\frac{d(e^u)}{du} = e^u, \text{ and } \frac{d(-x^2)}{dx} = -2x.$$

Plugging in,

$$\frac{dv}{dx} = e^u(-2x) = e^{-x^2}(-2x) = -2xe^{-x^2}.$$

Thus,

$$\frac{d}{dx}(x^2 e^{-x^2}) = 2xe^{-x^2} + x^2(-2xe^{-x^2}) = 2xe^{-x^2} - 2x^3 e^{-x^2} = \boxed{-2x(x^2 - 1)e^{-x^2}}.$$

Solution (g) Let u equal $\ln(y)$. Then, using rules of logarithms,

$$u = \ln(y) = \ln\left(\frac{x^2+3}{x+5}\right)^{1/5} = \frac{1}{5}\ln\left(\frac{x^2+3}{x+5}\right) = \frac{1}{5}\ln(x^2+3) - \frac{1}{5}\ln(x+5).$$

Thus,

$$\frac{du}{dx} = \frac{1}{5}\frac{d}{dx}(\ln(x^2+3)) - \frac{1}{5}\frac{d}{dx}(\ln(x+5)).$$

Let v equal $x^2 + 3$. By the chain rule,

$$\frac{d\ln(v)}{dx} = \frac{d\ln(v)}{dv}\frac{dv}{dx} = \frac{1}{v}(2x) = \frac{2x}{x^2+3}.$$

Let w equal $x + 5$. By the chain rule,

$$\frac{d\ln(w)}{dx} = \frac{d\ln(w)}{dw}\frac{dw}{dx} = \frac{1}{w}(1) = \frac{1}{x+5}.$$

Thus,

$$\frac{du}{dx} = \frac{2x}{5(x^2+3)} - \frac{1}{5(x+5)}.$$

On the other hand,

$$\frac{du}{dx} = \frac{d\ln(y)}{dx} = \frac{1}{y}\frac{dy}{dx}.$$

Therefore,

$$\frac{dy}{dx} = y\frac{du}{dx} = \boxed{\sqrt[5]{\frac{x^2+3}{x+5}}\left(\frac{2x}{5(x^2+3)} - \frac{1}{5(x+5)}\right).}$$

Solution (h) By the double-angle formula,

$$\cos(2\phi) = (\cos(\phi))^2 - (\sin(\phi))^2 = 2(\cos(\phi))^2 - [(\cos(\phi))^2 + (\sin(\phi))^2] = 2(\cos(\phi))^2 - 1.$$

Substituting 2θ for ϕ gives,

$$\cos(4\theta) = 2(\cos(2\theta))^2 - 1.$$

Subsituting in $\cos(2\theta) = 2(\cos(\theta))^2 - 1$ gives,

$$\begin{aligned}\cos(4\theta) &= 2[2(\cos(\theta))^2 - 1]^2 - 1 = 2[4(\cos(\theta))^4 - 4(\cos(\theta))^2 + 1] - 1 \\ &= \boxed{8(\cos(\theta))^4 - 8(\cos(\theta))^2 + 1.}\end{aligned}$$

Solution (i) First of all, $\ln(x^2)$ equals $2\ln(x)$. Thus,

$$y = \sin(2\ln(x)).$$

Let u equal $2\ln(x)$. Thus y equals $\sin(u)$. By the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

Of course,

$$\frac{du}{dx} = \frac{d(2\ln(x))}{dx} = \frac{2}{x}.$$

Also,

$$\frac{d\sin(u)}{du} = \cos(u).$$

Thus,

$$\frac{dy}{dx} = \cos(u)\frac{2}{x} = \boxed{2\cos(2\ln(x))/x.}$$

Solution (j) Writing the functions out in terms of $\sin(x)$ and $\cos(x)$,

$$y = \left(\frac{\cos(x)}{\sin(x)} + \frac{1}{\sin(x)}\right)^2 = \left(\frac{\cos(x)+1}{\sin(x)}\right)^2.$$

Let u equal $(\cos(x)+1)/\sin(x)$. Then $y = u^2$. By the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 2u\frac{du}{dx}.$$

Let v equal $\cos(x)+1$ and let w equal $\sin(x)$. Then $u = v/w$. By the quotient rule,

$$\frac{du}{dx} = \frac{1}{w^2}\left(\frac{dv}{dx}w - v\frac{dw}{dx}\right).$$

Also, dv/dx equals $-\sin(x)$ and dw/dx equals $\cos(x)$. Thus,

$$\begin{aligned}\tfrac{du}{dx} &= \tfrac{1}{(\sin(x))^2}((-\sin(x))\sin(x) - (\cos(x)+1)\cos(x)) \\ &= \tfrac{1}{(\sin(x))^2}(-(\sin(x))^2 - (\cos(x))^2 - \cos(x)) = \tfrac{-(\cos(x)+1)}{(\sin(x))^2}.\end{aligned}$$

Therefore,

$$\frac{dy}{dx} = \frac{2(\cos(x)+1)}{\sin(x)}\frac{-(\cos(x)+1)}{(\sin(x))^2} =$$

$$\boxed{-2(\cos(x)+1)^2/(\sin(x))^3.}$$

Part II(30 points)

Problem 1(5 points) Find the equation of the tangent line to the graph of $y = e^{571x}$ containing the point $(102\pi, 0)$. (This is *not* a point on the graph; it is a point on the tangent line.)

Solution to Problem 1 Denote 571 by the symbol n. Denote 102π by the symbol a. The derivative of e^{nx} equals ne^{nx}. Thus the slope of the tangent line to $y = e^{nx}$ at the point (b, e^{nb}) equals ne^{nb}. So the equation of the tangent line to $y = e^{nx}$ at (b, e^{nb}) is,

$$y = ne^{nb}(x-b) + e^{nb} = ne^{nb}x - (nb-1)e^{nb}.$$

If $(a, 0)$ is contained in this line, then the equation holds for $x = a$ and $y = 0$,

$$0 = ne^{nb}a - (nb-1)e^{nb} = (na - nb + 1)e^{nb}.$$

Since e^{nb} is not zero, dividing by e^{nb} gives,

$$na - nb + 1 = 0.$$

This can be solved to determine the one unknown in the equation, b:

$$b = (na + 1)/n.$$

Substituting this in gives the equation of the tangent line to $y = e^{nx}$ containing $(a, 0)$,

$$\boxed{y = ne^{(na+1)}x - nae^{(na+1)}.}$$

Problem 2(5 points)

(a)(2 points) What does the chain rule say if $y = x^a$ and $u = y^b$? The constants a and b are fractions.

Solution to (a) First of all, u equals y^b, which equals $(x^a)^b$. By the rules for exponents, this equals x^{ab}. According to the chain rule,

$$\frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} = (by^{b-1})(ax^{a-1}).$$

Substituting in $y = x^a$ gives,

$$\frac{du}{dx} = (b(x^a)^{b-1})(ax^{a-1}).$$

Using the rules for exponents, this equals,

$$\frac{du}{dx} = (bx^{a(b-1)})(ax^{a-1}) = abx^{ab-a}x^{a-1} = abx^{ab-1}.$$

This is precisely what the chain rule should give, since, setting $c = ab$,

$$\frac{d(x^c)}{dx} = cx^{c-1} = abx^{ab-1}.$$

(b)(3 points) Using the chain rule, give a very short explanation of the formula from Problem 3, Part II of Problem Set 1.

Solution to (b) Let u equal ax and let y equal $f(u)$. Then y equals $f(ax)$, which is $g(x)$. Thus $g'(x)$ equals dy/dx. By the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = f'(u)(ax)' = f'(ax)(a)$$

$$= \boxed{af'(ax).}$$

Problem 3(10 points) A bank offers *savings accounts* and *loans*. For an initial deposit of A dollars in a savings account with continuously compounded interest at an annual rate a, after t years the bank owes the customer $A(1+a)^t$ dollars (neglecting fees). For an initial loan of B dollars with continuously compounded interest at an annual rate b, after t years the customer owes the bank $B(1+b)^t$ dollars (neglecting fees). To make a profit, the bank sets rate b, the interest rate for loans, higher than rate a, the interest rate for savings. To simplify computations, introduce $\alpha = \ln(1+a)$ and $\beta = \ln(1+b)$.

Customer 1 deposits A dollars in a savings account. The bank immediately loans a smaller amount of B dollars to Customer 2. After t years, the bank's net gain from the two transactions together is,

$$G(t) = Be^{\beta t} - Ae^{\alpha t}. \tag{1}$$

In the long run, which is to say, when t is very large, $G(t)$ is positive and the bank has made a gain. However, for t small, $G(t)$ is negative and the bank has a net liability,

$$L(t) = -G(t) = Ae^{\alpha t} - Be^{\beta t}. \tag{2}$$

The liability for the savings account alone is,

$$M(t) = Ae^{\alpha t}. \tag{3}$$

In these equations, A, B, α and β are positive constants, and t is the independent variable.

(a)(5 points) Find the moment $t = T$ when the derivative $L'(T)$ equals 0. Assume that αA is greater than βB. Also, leave your answer in the form,

$$e^{(\beta-\alpha)T} = \text{something}.$$

Remark. After Lecture 10, we will learn that T is the moment when $L(t)$ has its largest value. In other words, if at time t Customer 1 withdraws all money, and Customer 2 repays all money, the bank loses the maximum amount when $t = T$.

Solution to (a) Using the chain rule,

$$L'(t) = A(e^{\alpha t})' - B(e^{\beta t})' = A(\alpha e^{\alpha t}) - B(\beta e^{\beta t}) = \alpha Ae^{\alpha t} - \beta Be^{\beta t}.$$

By definition of T, $L'(T)$ equals 0. Thus,

$$\beta Be^{\beta T} = \alpha Ae^{\alpha T}.$$

Dividing each side of the equation by $\beta Be^{\alpha T}$ gives,

$$e^{\beta T}/e^{\alpha T} = \frac{\alpha A}{\beta B}.$$

Using rules of exponents, this is,

$$e^{(\beta-\alpha)T} = \boxed{(\alpha A)/(\beta B).}$$

It is worth remarking that to make T small, the bank may maximize the fraction B/A of money loaned to money deposited and the bank may maximize the fraction β/α (although if β is too high or α too low, customers are discouraged from using the bank).

(b)(5 points) Consider the ratio $L(t)/M(t)$. Using your answer to **(a)**, determine $L(T)/M(T)$. Simplify your answer as much as possible. How does this ratio depend on the amounts A and B?

Solution to (b) The ratio $L(t)/M(T)$ equals,

$$\frac{L(t)}{M(t)} = \frac{Ae^{\alpha t} - Be^{\beta t}}{A^{\alpha t}} = 1 - \frac{B}{A}e^{(\beta-\alpha)t}.$$

By the formula from the **Solution to (a)**, $e^{(\beta-\alpha)T}$ equals $(\alpha A)/(\beta B)$. Plugging this in,

$$\frac{L(T)}{M(T)} = 1 - \frac{B}{A}\frac{\alpha A}{\beta B} = \boxed{1 - (\alpha/\beta).}$$

In particular, this is independent of A and B.

Remark. From the formulas, the bank's strategy is clear. First, adjust the ratio β/α to the highest level allowed by law and compatible with the market's demands. Then, for fixed α and β, the ratio $L(T)/M(T)$ is independent of A and B. So the maximal liability $L(T)$ is proportional to $M(T)$. Since $M(T)$ is an increasing function, the strategy is to minimize T, by maximizing the ratio B/A. (Of course this ratio will always be less than 1, since some fraction of all capital goes to the federal reserve, some fraction is used to cover operating expenses, etc.)

Problem 4(10 points) Let A, β, ω and t_0 be positive constants. Let $f(t)$ be the function,

$$f(t) = Ae^{-\beta t}\cos(\omega(t - t_0)).$$

(a)(5 points) Compute $f'(t)$ and $f''(t)$. Simplify your answer as much as possible.

Solution to (a) Let s equal $t - t_0$. Then $f(t) = g(s)$, where,

$$g(s) = Be^{-\beta s}\cos(\omega s),$$

and $B = Ae^{-\beta t_0}$. The derivative ds/dt equals 1. Thus, according to the chain rule,

$$\frac{df}{dt} = \frac{dg}{ds}.$$

Using the product rule and the chain rule, this equals,

$$\begin{array}{c} \frac{dg}{ds} = B(e^{-\beta s})'\cos(\omega s) + Be^{-\beta s}(\cos(\omega s))' = \\ B(-\beta e^{-\beta s})\cos(\omega s) + Be^{-\beta s}(-\omega\sin(\omega s)) = \\ -B\beta e^{-\beta s}\cos(\omega s) - B\omega e^{-\beta s}\sin(\omega s). \end{array}$$

In particular, when B equals 1, this gives,

$$\frac{d}{ds}(e^{-\beta s}\cos(\omega s)) = -\beta e^{-\beta s}\cos(\omega s) - \omega e^{-\beta s}\sin(\omega s)).$$

The second derivative $g''(s)$ involves the derivative of $e^{-\beta s}\cos(\omega s)$, but it is also involves the derivative of $e^{-\beta s}\sin(\omega s)$. Using the chain rule and the product rule,

$$\begin{array}{c}\frac{d}{ds}(e^{-\beta s}\sin(\omega s)) = (e^{-\beta s})'\sin(\omega s) + e^{-\beta s}(\sin(\omega s))' = \\ -\beta e^{-\beta s}\sin(\omega s) + \omega e^{-\beta s}\cos(\omega s).\end{array}$$

As above,

$$\frac{d^2 f}{dt^2} = \frac{d^2 g}{ds^2}.$$

This is,

$$\begin{array}{c}\frac{d}{ds}(\frac{dg}{ds}) = -B\beta[e^{-\beta s}\cos(\omega s)]' - B\omega[e^{-\beta s}\sin(\omega s)]' = \\ -B\beta[-\beta e^{-\beta s}\cos(\omega s) - \omega e^{-\beta s}\sin(\omega s)] - B\omega[-\beta e^{-\beta s}\sin(\omega s) + \omega e^{-\beta s}\cos(\omega s)] = \\ (B\beta^2 - B\omega^2)e^{-\beta s}\cos(\omega s) + (B\beta\omega + B\beta\omega)e^{-\beta s}\sin(\omega s) = \\ B(\beta^2 - \omega^2)e^{-\beta s}\cos(\omega s) + 2B\beta\omega e^{-\beta s}\sin(\omega s).\end{array}$$

Back-substituting $s = t - t_0$ and $Be^{-\beta s} = Ae^{-\beta t}$ gives,

$$f'(t) = \boxed{-A\beta e^{-\beta t}\cos(\omega(t-t_0)) - A'\omega e^{-\beta t}\sin(\omega(t-t_0)),}$$

and,

$$f''(t) = \boxed{A(\beta^2-\omega^2)e^{-\beta t}\cos(\omega(t-t_0)) + 2A\beta\omega e^{-\beta t}\sin(\omega(t-t_0)).}$$

(b)(5 points) Using your answer to **(a)**, find nonzero constants c_0, c_1 and c_2 for which the function

$$c_2 f''(t) + c_1 f'(t) + c_0 f(t),$$

always equals 0.

Solution to (b) From the **Solution to (a)**, $f'(t)+\beta f(t)$ equals $-Ae^{-\beta t}(\omega\sin(\omega(t-t_0)))$. Plugging this into the formula for $f''(t)$ gives,

$$f''(t) = (\beta^2-\omega^2)f(t) - 2\beta(f'(t)+\beta(f)) = -2\beta f' - (\beta^2+\omega^2)f(t).$$

Simplifying gives,

$$f''(t) + 2\beta f'(t) + (\omega^2+\beta^2)f(t) = 0.$$

In fact, every solution is of the form,

$$\boxed{c_2 = c, c_1 = 2\beta c, c_0 = \omega^2+\beta^2,}$$

for some nonzero c.