18.01 Calculus **Jason Starr**
Due by 2:00pm sharp **Fall 2005**
Friday, Sept. 16, 2005

Solutions to Problem Set 1

Part I/Part II

Part I(20 points)

(a)	(2 points)	p. 57,	Section 2.2,	Problem 1	
(b)	(2 points)	p. 62,	Section 2.3,	Problem 1	
(c)	(2 points)	p. 62,	Section 2.3,	Problem 28	
(d)	(2 points)	p. 62,	Section 2.3,	Problem 36	You may assume $x > 0$.
(e)	(2 points)	p. 68,	Section 2.4,	Problem 8	"Speed" is different from "velocity".
(f)	(2 points)	p. 73,	Section 2.5,	Problem 4	
(g)	(2 points)	p. 73,	Section 2.5,	Problem 6	
(h)	(2 points)	p. 73,	Section 2.5,	Problem 14	
(i)	(2 points)	p. 87,	Section 3.1,	Problem 5	
(j)	(2 points)	p. 91,	Section 3.2,	Problem 30	

Solution (a) By Equation (4) on p. 54, the slope of the tangent line to the parabola at (x_0, y_0) is $2x_0$. Thus the equation of the tangent line is,

$$(y - y_0) = 2x_0(x - x_0), \text{ or equivalently } y = 2x_0x - x_0^2.$$

(a). At the point $(x_0, y_0) = (-2, 4)$, the tangent line is,

$$\boxed{y = -4x - 4.}$$

(b). Since the slope is $2x_0$, the slope is 8 when x_0 equals 4. Plugging in, y_0 equals $x_0^2 = (4)^2$, which is 16. Thus the tangent line with slope 8 is,

$$\boxed{y = 8x - 16.}$$

(c). If x_0 is zero, the tangent line has equation $y = 0$, i.e., the tangent line is the x-axis. Thus the tangent line does not have a well-defined x-intercept. Therefore assume that x_0 is nonzero. The x-intercept of the tangent line is the value x_1 such that y is 0. Plugging in, this gives the equation,

$$0 = 2x_0x_1 - x_0^2.$$

Simplifying, this is $2x_0x_1 = x_0^2$. Since x_0 is nonzero by hypothesis, also $2x_0$ is nonzero. Dividing both side by $2x_0$ gives the equation $x_1 = x_0/2$. Thus the x-intercept x_1 equals 2 if and only if x_0 equals $2x_1 = 2 \times 2 = 4$. As computed in (b), the equation of the tangent line is,

$$\boxed{y = 8x - 16.}$$

Solution (b) Since $f(x)$ equals $ax^2 + bx + c$, substituting $x + \Delta x$ for x, $f(x + \Delta x)$ equals,

$$a(x + \Delta x)^2 + b(x + \Delta x) + c.$$

Expanding the square, this is,

$$a(x^2 + 2x\Delta x + (\Delta x)^2) + b(x + \Delta x) + c.$$

Therefore $f(x + \Delta x) - f(x)$ equals,

$$[a(x^2 + 2x\Delta x + (\Delta x)^2) + b(x + \Delta x) + c] - [ax^2 + bx + c].$$

Cancelling like terms, namely ax^2, bx and c, this simplifies to,

$$2ax\Delta x + a(\Delta x)^2 + b\Delta x.$$

Separating the common factor Δx from these terms, this simplifies to,

$$f(x + \Delta x) - f(x) = (2ax + a\Delta x + b)\Delta x.$$

This complete Step 1.

Because of the common factor Δx, the difference quotient is,

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = 2ax + a\Delta x + b,$$

for Δx nonzero. This completes Step 2.

Holding a, b and x constant and allowing Δx to vary, the expression $2ax + a\Delta x + b$ is a linear function in Δx; perhaps this is easier to see if it is written $a\Delta x + (2ax + b)$. A linear function is continuous. Thus to compute the limit as Δx approaches 0, it suffices to substitute in Δx equals 0. Therefore,

$$f'(x) = \lim_{\Delta x \to 0}[a\Delta x + (2ax + b)] = a0 + (2ax + b),$$

which simplifies to $2ax + b$. Therefore the derivative of $ax^2 + bx + c$ is,

$$\boxed{f'(x) = 2ax + b.}$$

Solution (c) The function is defined when $3x+2$ is nonzero, i.e., when x is not $-2/3$. The function is undefined with x equals $-2/3$. Therefore assume that x is not $-2/3$.

Substituting $x + \Delta x$ for x gives,

$$f(x + \Delta x) = \frac{1}{3(x + \Delta x) + 2}.$$

To compute the difference,

$$f(x + \Delta x) - f(x) = \frac{1}{3(x + \Delta x) + 2} - \frac{1}{3x + 2},$$

we express both fractions with the common denominator $(3(x + \Delta x) + 2)(3x + 2)$,

$$\left[\frac{1}{3(x + \Delta x) + 2} \times \frac{3x + 2}{3x + 2}\right] - \left[\frac{1}{3x + 2} \times \frac{3(x + \Delta x) + 2}{3(x + \Delta x) + 2}\right] =$$

$$\frac{3x + 2}{(3(x + \Delta x) + 2)(3x + 2)} - \frac{3(x + \Delta x) + 2}{(3(x + \Delta x) + 2)(3x + 2)}.$$

This simplifies to,

$$\frac{(3x + 2) - (3(x + \Delta x) + 2)}{(3(x + \Delta x) + 2)(3x + 2)}.$$

Cancelling the like terms $3x$ and 2, this simplifies to,

$$f(x + \Delta x) - f(x) = \frac{-3\Delta x}{(3(x + \Delta x) + 2)(3x + 2)}.$$

This completes Step 1.

Because of the factor Δx in the numerator of $f(x + \Delta x) - f(x)$, the difference quotient is,

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{-3}{(3(x + \Delta x) + 2)(3x + 2)},$$

for Δx nonzero. This completes Step 2.

Considered as a function of Δx, is the expression $(-3)/(3x+2+3\Delta x)(3x+2)$ defined and continuous at $\Delta x = 0$? The only values of Δx where the expression is undefined or discontinuous are the values where the denominator equals 0.

By hypothesis, x is not $-2/3$, and thus $3x + 2$ is not zero. Therefore the denominator is 0 if and only if $3x + 2 + 3\Delta x$ is 0. Thus the function $(-3)/(3x + 2 + 3\Delta x)(3x + 2)$ has a single infinite discontinuity when Δx equals $-x - 2/3$. Again using the hypothesis that x is not $-2/3$, $-x - 2/3$ does not equal 0. In other words, there is a single point where the function is undefined and discontinuous, but this point is different from $\Delta x = 0$. Therefore $(-3)/(3x + 2 + 3\Delta x)(3x + 2)$ *is* defined and continuous at $\Delta x = 0$. So the limit can be computed by substituting in 0 for Δx. Therefore the derivative of $f(x) = 1/(3x + 2)$ is,

$$f'(x) = \lim_{\Delta x \to 0} \frac{-3}{(3x + 2 + 3\Delta x)(3x + 2)} = \boxed{-3/(3x + 2)^2.}$$

Solution (d) The function is defined when x is nonnegative and undefined when x is negative. Therefore assume that x is nonnegative, $x \geq 0$.

Substituting $x + \Delta x$ for x gives,

$$f(x + \Delta x) = \sqrt{2(x + \Delta x)}.$$

Please note this is defined if and only if $x + \Delta x$ is nonnegative, i.e., $\Delta x \geq -x$. Also, as always, Δx is nonzero.

To compute the difference,

$$f(x + \Delta x) - f(x) = \sqrt{2(x + \Delta x)} - \sqrt{2x},$$

multiply and divide by the sum $\sqrt{2(x + \Delta x)} + \sqrt{2x}$,

$$f(x + \Delta x) - f(x) = (\sqrt{2(x + \Delta x)} - \sqrt{2x}) \times \frac{\sqrt{2(x + \Delta x)} + \sqrt{2x}}{\sqrt{2(x + \Delta x)} + \sqrt{2x}}.$$

Although at first glance this seems to make the expression more complicated, in fact the expression now simplifies. The numerator is of the form $(a - b)(a + b)$ for $a = \sqrt{2(x + \Delta x)}$ and $b = \sqrt{2x}$. But $(a - b)(a + b)$ simplifies to a "difference of squares", $a^2 - b^2$. Thus the expression simplifies to,

$$\frac{(\sqrt{2(x + \Delta x)})^2 - (\sqrt{2x})^2}{\sqrt{2(x + \Delta x)} + \sqrt{2x}} = \frac{2(x + \Delta x) - (2x)}{\sqrt{2(x + \Delta x)} + \sqrt{2x}}.$$

Cancelling the like term $2x$, this simplifies to,

$$f(x + \Delta x) - f(x) = \frac{2\Delta x}{\sqrt{2(x + \Delta x)} + \sqrt{2x}}.$$

This completes Step 1.

Because of the factor Δx in the numerator of $f(x + \Delta x) - f(x)$, the difference quotient is,

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{2}{\sqrt{2(x + \Delta x)} + \sqrt{2x}},$$

for Δx nonzero and satisfying $\Delta x \geq -x$. This completes Step 2.

There are 2 cases depending on whether x is positive or zero. First consider the case that x is zero. Then the difference quotient is,

$$\frac{2}{\sqrt{2\Delta x}}.$$

This expression has an infinite discontinuity as Δx approaches 0. Therefore the limit is undefined. Since the derivative is the limit of the difference quotient, $f'(x)$ is undefined for $x = 0$.

Next consider the case that x is positive. Considered as a function of Δx, for $\Delta x \geq -x$, the expression $2/(\sqrt{2(x+\Delta x)}+\sqrt{2x})$ is defined and continuous as long as the denominator is nonzero. Since $\sqrt{2x}$ is positive and $\sqrt{2(x+\Delta x)}$ is nonnegative, the sum $\sqrt{2(x+\Delta x)}+\sqrt{2x}$ is positive. Therefore the expression is defined and continuous at $\Delta x = 0$. So the limit can be computed by substituting in 0 for Δx. Therefore the derivative of $f(x) = \sqrt{2x}$ for $x > 0$ is,

$$f'(x) = \lim_{\Delta x \to 0} \frac{2}{\sqrt{2(x+\Delta x)}+\sqrt{2x}} = \frac{2}{2\sqrt{2x}} = \frac{1}{\sqrt{2x}}.$$

To summarize, $f'(x)$ is undefined for $x = 0$ and $f'(x) = 1/\sqrt{2x}$ for $x > 0$.

Solution (e) The velocity of the first particle is,

$$v_1(t) = s_1'(t) = 2t - 6,$$

and the velocity of the second particle is,

$$v_2(t) = s_2'(t) = -4t + 9,$$

using the solution of Problem 1 of Section 2.3. Therefore the speed of the first particle is,

$$|v_1(t)| = |2t - 6| = \begin{cases} 6 - 2t, & t \leq 3 \\ 2t - 6, & t > 3. \end{cases}$$

and the speed of the second particle is,

$$|v_2(t)| = |-4t + 9| = \begin{cases} 9 - 4t, & t \leq 9/4 \\ 4t - 9, & t > 9/4. \end{cases}$$

(a). There are 3 cases to consider: $0 \leq t \leq 9/4$, $9/4 < t \leq 3$ and $t > 3$. In the first case, $|v_1(t)|$ equals $|v_2(t)|$ if and only if,

$$(6 - 2t = 9 - 4t) \text{ if and only if } (2t = 3) \text{ if and only if } (t = 3/2).$$

So for $0 \leq t \leq 9/4$, the speeds are equal for precisely one moment, $t = 3/2$. At this time, both speeds equal 3. Note, however, the velocity of the first particle is -3 and the velocity of the second particle is $+3$, i.e., the velocities are not equal.

In the second case, $|v_1(t)|$ equals $|v_2(t)|$ if and only if,

$$(6 - 2t = 4t - 9) \text{ if and only if } (6t = 15)\text{if and only if } (t = 15/6).$$

Note that $15/6 = 5/2 = 2\frac{1}{2}$ is between $9/4 = 2\frac{1}{4}$ and 3. So, for $9/4 < t \leq 3$, the speeds are equal for precisely one moment, $t = 5/2$. At this time, both speeds equal 1.

In the third case, $|v_1(t)|$ equals $|v_2(t)|$ if and only if,

$$(2t - 6 = 4t - 9) \text{ if and only if } (2t = 3) \text{ if and only if } (t = 3/2).$$

However, $3/2 = 1\frac{1}{2}$ is less than 3. So for $t > 9/4$, the particles never have the same speed. In summary, for $t \geq 0$ the two particles have equal speed at precisely two moments, $\boxed{t = 3/2 \text{ and } t = 5/2.}$

(b). The moment when the two particles have the same position is the solution of the equation,

$$s_1(t) = s_2(t), \text{ or equivalently}$$
$$t^2 - 6t = -2t^2 + 9t, \text{ or equivalently}$$
$$3t^2 = 15t.$$

The two solutions of this quadratic equation are $t = 0$ and $t = 5$.

Now, $v_1(0)$ equals $2 \times 0 - 6 = -6$ and $v_2(0)$ equals $-4 \times 0 + 9 = 9$. Also, $v_1(5)$ equals $2 \times 5 - 6 = 4$ and $v_2(5)$ equals $-4 \times 5 + 9 = -11$. Thus, for $t = 0$, the particles have velocities, $\boxed{v_1 = -6 \text{ and } v_2 = 9.}$ And, for $t = 5$, the particles have velocities $\boxed{v_1 = 4 \text{ and } v_2 = -11.}$

Solution (f) The expression $6/(2x - 4)$ has an infinite discontinuity when the denominator equals 0. The denominator is 0 when $2x - 4 = 0$, or equivalently, $x = 2$. Therefore the limit,

$$\lim_{x \to 2} \frac{6}{2x - 4},$$

is $\boxed{\text{undefined.}}$

Solution (g) The expression $(x^2 + 3x)/(x^2 - x + 3)$ is defined and continuous so long as the denominator is nonzero. Plugging in 3 for x, the denominator equals,

$$(3)^2 - (3) + 3 = 9,$$

when $x = 3$. Since the denominator is nonzero, the limit exists and equals,

$$\lim_{x \to 3} \frac{x^2 + 3x}{x^2 - x + 3} = \frac{(3)^2 + 3(3)}{(3)^2 - (3) + 3} = \frac{18}{9} = \boxed{2.}$$

Solution (h) The expression $(x - 4)/(\sqrt{x} - 2)$ is undefined when $x = 4$, since the denominator is 0. However, by the same "difference of squares" technique from **Solution (d)**,

$$\frac{1}{\sqrt{x} - 2} = \frac{1}{\sqrt{x} - 2} \times \frac{\sqrt{x} + 2}{\sqrt{x} + 2} = \frac{\sqrt{x} + 2}{x - 4},$$

for x nonnegative and not 4. Therefore, for x nonnegative and not 4,

$$\frac{x - 4}{\sqrt{x} - 2} = \sqrt{x} + 2.$$

The expression $\sqrt{x}+2$ is defined and continuous for all nonnegative x. Therefore the limit is obtained by plugging in 4 for x;

$$\lim_{x\to 4} \frac{x-4}{\sqrt{x}-2} = \lim_{x\to 4}(\sqrt{x}+2) = \sqrt{4}+2 = \boxed{4.}$$

Solution (i) By Rules 1–4 on pp. 84–85, the derivative is,

$$\begin{array}{rcll} y' & = & (3x^2)' + (-5x)' + (2)' & \text{(Rule 4)} \\ & = & 3(x^2)' + (-5)(x)' + 2(1)' & \text{(Rule 3)} \\ & = & 3(2x) + (-5)(1) + 2(0) & \text{(Rule 2 and Rule 1)} \\ & = & 6x - 5. & \end{array}$$

Of course this is also a special case of Problem 1 from Section 2.3. At $x = 2$, the derivative is $y' = 6(2) - 5 = 7$. Therefore the equation of the tangent line is,

$$(y-4) = 7(x-2) \text{ or equivalently } y = \boxed{7x - 10.}$$

Solution (j) In the first method, the fraction is simplified to,

$$f(x) = \frac{2x + 6x^4 - 2x^6}{x^5} = 2x^{-4} + 6x^{-1} - 2x.$$

Using Equation (3) on p. 90, the derivative is,

$$f'(x) = 2(x^{-4})' + 6(x^{-1})' - 2(x)' = 2(-4x^{-5}) + 6(-x^{-2}) - 2(1) = -8x^{-5} - 6x^{-2} - 2.$$

Clearing denominators, the derivative is,

$$\boxed{f'(x) = (-8 - 6x^3 - 2x^5)/x^5.}$$

In the second method, expressing $f(x)$ as a quotient,

$$f(x) = g(x)/h(x), \quad g(x) = 2x + 6x^4 - 2x^6, \ h(x) = x^5,$$

the quotient rule gives,

$$f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{h(x)^2}.$$

Using Section 3.1,

$$g'(x) = 2 + 24x^3 - 12x^5, \ h'(x) = 5x^4.$$

Therefore the quotient rule gives,

$$f'(x) = \frac{(2 + 24x^3 - 12x^5)(x^5) - (2x + 6x^4 - 2x^6)(5x^4)}{x^{10}}.$$

Expanding and simplifying, the numerator equals,

$$(2x^5 + 24x^8 - 12x^{10}) - (10x^5 + 30x^8 - 10x^{10}) = -8x^5 - 6x^8 - 2x^{10}.$$

Thus the quotient rule gives,

$$f'(x) = \frac{-8x^5 - 6x^8 - 2x^{10}}{x^{10}}.$$

Factoring x^5 from numerator and denominator, this is,

$$f'(x) = \boxed{(-8 - 6x^3 - 2x^5)/x^5,}$$

just as in the first method.

Part II(30 points)

Problem 1(15 points) The derivative of $f(x) = 1/x$ is $f'(x) = -1/x^2$ (for x nonzero).

(a)(5 points) Show that for the tangent line to the graph of $f(x)$ at (x_0, y_0), the x-intercept of the line is $2x_0$ and the y-intercept of the line is $2y_0$.

Solution to (a) The equation of the tangent line is,

$$(y - y_0) = \frac{-1}{x_0^2}(x - x_0).$$

The x-intercept is the unique value $x = x_1$ for which $y = 0$. Plugging in $x = x_1$ and $y = 0$ gives the equation,

$$-y_0 = \frac{-1}{x_0^2}(x_1 - x_0).$$

Simplifying, this gives,

$$x_1 = x_0 + x_0^2 y_0.$$

Since $x_0 y_0$ equals 1, this simplifies to,

$$x_1 = x_0 + x_0(x_0 y_0) = x_0 + x_0(1) = \boxed{2x_0.}$$

Similarly, the y-intercept is the unique value $y = y_1$ for which $x = 0$. Plugging in $x = 0$ and $y = y_1$ gives the equation,

$$y_1 - y_0 = \frac{-1}{x_0^2}(-x_0) = \frac{1}{x_0}.$$

Since $1/x_0$ equals y_0, the equation is,

$$y_1 - y_0 = y_0.$$

Solving, this gives,

$$y_1 = \boxed{2y_0.}$$

(b)(5 points) **Part (a)** implies the following: For every pair of real numbers (x_1, y_1), there is a tangent line to the graph of $f(x)$ with x-intercept x_1 and y-intercept y_1 if and only if x_1y_1 equals 4. You may use this fact freely.

Let (a, b) be a point such that ab is nonzero and less than 1 (possibly negative). Show that a line L with x-intercept x_1 and y-intercept y_1 is a tangent line to the graph of $f(x)$ containing (a, b) if and only if x_1 satisfies,

$$bx_1^2 - 4x_1 + 4a = 0,$$

and y_1 satisfies,

$$ay_1^2 - 4y_1 + 4b = 0.$$

Hint. Using Equations 1–5 on pp. 11–12 of the textbook, deduce that L contains (a, b) if and only if $bx_1 + ay_1$ equals x_1y_1. Then use the fact above to eliminate one of x_1 or y_1 from this equation, and simplify.

Solution to (b) The equation of the line L with x-intercept x_1 and y-intercept y_1 is,

$$x_1y + y_1x = x_1y_1.$$

Therefore (a, b) is on L if and only if,

$$bx_1 + ay_1 = x_1y_1. \tag{1}$$

First, substituting in $y_1 = 4/x_1$ to Equation 1 gives,

$$bx_1 + \frac{4a}{x_1} = 4.$$

Multiplying both sides by x_1 and simplifying gives,

$$\boxed{bx_1^2 - 4x_1 + 4a = 0.}$$

Next, substituting in $x_1 = 4/y_1$ to Equation 1 gives,

$$\frac{4b}{y_1} + ay_1 = 4.$$

Multiplying both sides by y_1 and simplifying gives,

$$\boxed{ay_1^2 - 4y_1 + 4b = 0.}$$

(c)(5 points) Using the quadratic formula and **Part (b)**, which you may now use freely, write down the equations of the 2 tangent lines to the graph of $f(x)$ containing the point $(a, b) = (5, -3)$.

Solution to (c) Plugging in $a = 5$ and $b = -3$ to the equation $bx_1^2 - 4x_1 + 4a = 0$ gives,

$$-3x_1^2 - 4x_1 + 20 = 0.$$

By the quadratic formula, the solutions are,

$$x_1 = \frac{4}{2(-3)} \pm \frac{1}{2(-3)}\sqrt{(-4)^2 - 4(-3)(20)}.$$

Simplifying, this gives,

$$x_1 = \frac{-2}{3} \pm \frac{1}{-6}\sqrt{256} = \frac{-2}{3} \pm \frac{16}{6} = \frac{-2 \pm 8}{3}.$$

Thus the solutions are $x_1 = 6/3 = 2$ and $x_1 = -10/3$. Since $y_1 = 4/x_2$, the corresponding solutions of y_1 are $y_1 = 2$ and $y_1 = -6/5$.

Since the equation of L is $y_1 x + x_1 y = x_1 y_1$, the equations of the 2 tangent lines containing $(5, -3)$ are,

$$2x + 2y = 4$$
$$\tfrac{-6}{5}x + \tfrac{-10}{3}y = 4.$$

Simplifying, the equations of the 2 tangent lines containing $(5, -3)$ are,

$$x + y = 2 \text{ and } 9x + 25y = -30.$$

Problem 2(10 points) For a mass moving vertically under constant acceleration $-g$, the displacement function is,

$$x(t) = -gt^2/2 + v_0 t + x_0,$$

where x_0 is the displacement and v_0 is the instantaneous velocity at time $t = 0$.

A scientist uses a magnetic field to conduct an experiment simulating zero gravity. At time $t = 0$, the scientist drops a mass from a height of $10m$ with instantaneous velocity $v_0 = 0$ under constant acceleration $-10m/s^2$. When the mass drops below height $5m$, the field is switched on and the particle continues to move with new acceleration $0m/s^2$.

Assuming the displacement and velocity are continuous, determine the height and instantaneous velocity of the mass at time $t = 1.2s$. Show your work.

Solution to Problem 2 Before the field is switched on, the displacement function is,

$$x(t) = -10t^2/2 + 0t + 10 = -5t^2 + 10.$$

Differentiating, the velocity function is,

$$v(t) = x'(t) = -10t.$$

The time t_1 at which the field is switched on is the positive solution of the equation $x(t_1) = 5$. Plugging in and solving gives,

$$5 = -5t_1^2 + 10 \text{ or equivalently } 5t_1^2 = 5 \text{ or equivalently } t_1^2 = 1.$$

Therefore the field is activated at time $t_1 = 1s$.

The displacement function for $t > t_1$ is,

$$x(t) = g_1(t-t_1)^2/2 + v_1(t-t_1) + x_1,$$

where g_1 is the new acceleration, v_1 is the instantaneous velocity at time $t = t_1$, and x_1 is the displacement at time $t = t_1$. At time t_1, the displacement is $x_1 = 5$ and the instantaneous velocity is $v(t_1) = -10(1) = -10$. For $t > t_1$, the particle moves with acceleration $g_1 = 0$. Thus the displacement function is,

$$x(t) = 0(t-1)^2 + (-10)(t-1) + 5 = -10t + 15.$$

Plugging in $t = 1.2s$ gives,

$$x(1.2) = -10(1.2) + 15 = -12 + 15 = \boxed{3m.}$$

Problem 3(5 points) For a differentiable function $f(x)$ and a real number a, using the difference quotient definition of the derivative, show that the function $g(x) = f(ax)$ has derivative,

$$g'(x) = af'(ax).$$

Solution to Problem 3 Plugging in,

$$g(x+\Delta x) - g(x) = f(a(x+\Delta x)) - f(ax) = f(ax + a\Delta x) - f(ax).$$

Thus the difference quotient for $g(x)$ is,

$$\frac{g(x+\Delta x) - g(x)}{\Delta x} = \frac{f(ax+a\Delta x) - f(ax)}{\Delta x}.$$

By definition of the derivative,

$$g'(x) = \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}.$$

Now, for an expression $E(\Delta x)$ involving Δx,

$$\lim_{\Delta x \to \Delta x_0} E(\Delta x) = \lim_{a\Delta x \to a\Delta x_0} E(\Delta x).$$

Applying this to the limit above, and using that $a(0)$ equals 0,

$$g'(x) = \lim_{a\Delta x \to 0} \frac{f(ax+a\Delta x) - f(ax)}{\Delta x} = \lim_{a\Delta x \to 0} \frac{f(ax+a\Delta x) - f(ax)}{\Delta x} \times \frac{a}{a} =$$

$$a \lim_{a\Delta x \to 0} \frac{f(ax+a\Delta x) - f(ax)}{a\Delta x}.$$

Substituting $h = a\Delta x$, this is,

$$g'(x) = a \lim_{h \to 0} \frac{f(ax+h) - f(ax)}{h}.$$

This limit is precisely the derivative of $f(x)$ at the point ax. Therefore,

$$\boxed{g'(x) = af'(ax).}$$