Prob. 6.8

Assuming the material in a spherical rubber balloon can be modeled as linearly elastic with modulus *E* and Poisson's ratio ν = 0.5, show that the internal pressure *p* needed to expand the balloon varies with the radial expansion ratio $\lambda = r/r_0$ as

$$\frac{pr_0}{4Eb_0} = \frac{1}{\lambda_r^2} - \frac{1}{\lambda_r^3}$$

where b_0 is the initial wall thickness. Plot this function and determine its critical values.

The true stress as given by Eq. 6.1 is

$$\sigma_\theta = \sigma_\phi \equiv \sigma = \frac{pr}{2b}$$

Since the material is incompressible, the current wall thickness *b* is related to the original thickness b_o as

$$4\pi r^2 \cdot b = 4\pi r_0^2 \cdot b_0 \Rightarrow b = b_0\left(\frac{r_0}{r}\right)^2 = \frac{b_0}{\lambda_r^2}$$

The stress is then

$$\sigma = \frac{p}{2}\frac{r}{b_0}\lambda_r^2 = \frac{p}{2}\frac{r_o}{b_0}\lambda_r^3 \qquad (1)$$

The strain is

$$\varepsilon_\theta = \varepsilon_\phi \equiv \varepsilon = \frac{2\pi r - 2\pi r_0}{2\pi r_0} = \frac{r}{r_0} - 1 \equiv \lambda_r - 1 \qquad (2)$$

If the material is linearly elastic, the strain and stress are related as

$$\varepsilon_\phi = \frac{1}{E}\left[\sigma_\phi - \nu(\sigma_\theta + \sigma_r)\right] = \frac{\sigma}{E}(1-0.5) = \frac{\sigma}{2E} \qquad (3)$$

Using (1) and (2) in (3):

$$\lambda_r - 1 = \frac{1}{2E}\frac{p}{2}\frac{r_o}{b_0}\lambda_r^3 \Rightarrow \frac{pr_o}{4Eb_0} = \frac{\lambda_r - 1}{\lambda_r^3}$$

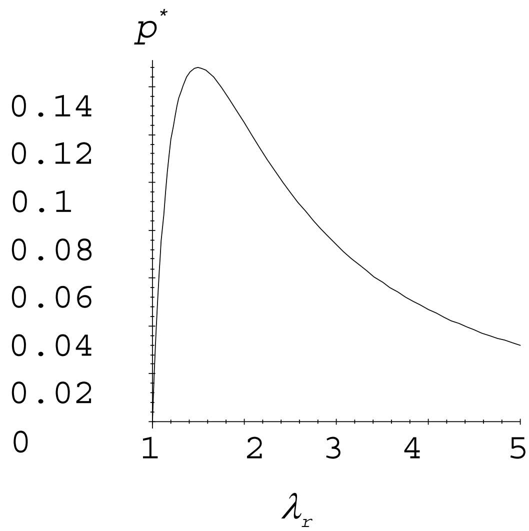
$$\frac{pr_o}{4Eb_0} = \frac{1}{\lambda_r^2} - \frac{1}{\lambda_r^3}$$

Plot:

```
pstar:=1/lambda[r]^2 - 1/lambda[r]^3;
```

$$pstar := \frac{1}{\lambda_r^{\,2}} - \frac{1}{\lambda_r^{\,3}}$$

```
plot(pstar,lambda[r]=1..5);
```



Determine λ_r at maximum pressure:

'lambda[r,max]'=solve(diff(pstar,lambda[r])=0,lambda[r]);

$$\lambda_{r,\,max} = \frac{3}{2}$$

The maximum normalized pressure is

Digits:=4;'pstar[max]'=evalf(subs(lambda[r]=3/2,pstar));

$$pstar_{max} = .1481$$

This maximum is commonly experienced as a yield-like phenomenon in blowing up a balloon. However, its origin is geometrical and not a function of the material.

Prob. 6.9

Repeat the previous problem, but using the given constitutive relation for rubber:

$${}_t\sigma_x = \frac{E}{3}\left(\lambda_x^2 - \frac{1}{\lambda_x^2\lambda_y^2}\right)$$

The circumfrential extension ratio is:

$$\lambda_\theta = \lambda_\phi = \frac{2\pi r}{2\pi r_0} = \lambda_r$$

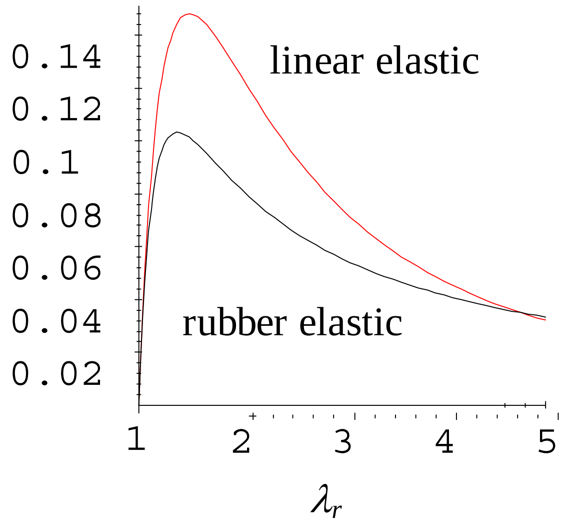
This is also both λ_x and λ_y in the given relation. From Eq. (1) of the previous solution we can write

$$\sigma = \frac{p}{2}\frac{r_o}{b_0}\lambda_r^3 = \frac{E}{3}\left(\lambda_r^2 - \frac{1}{\lambda_r^4}\right)$$

$$\boxed{\frac{pr_o}{4b_0E} = \frac{1}{6}\left(\frac{1}{\lambda_r} - \frac{1}{\lambda_r^7}\right)}$$

Plotting this along with the prevous result:

```
pstar1:= 1/lambda[r]^2 - 1/lambda[r]^3;
pstar2:= (1/6)*(1/lambda[r] - 1/lambda[r]^7);
plot({pstar1,pstar2},lambda[r]=1..5);
```



Extension at maximum pressure:

```
Digits:=4;'lambda[r,max]'=fsolve(diff(pstar2,lambda[r])=0,lambda[r]);
```

$$\lambda_{r,\, max} = -1.383$$

The maximum normalized pressure:

```
'pstar[max]'=evalf(subs(lambda[r]=1.383,pstar2));
```

$$pstar_{max} = .1033$$