PROFESSOR: Resume by going back to our one-dimensional body that has undergone some elastic deformation. And what I would like to do now is to distinguish between displacement of an object and fractional change of length, which turned out to be measured by the same thing, that thing that we're going to name strain when properly defined.

OK, here is our one-dimensional case. And we said that originally some point, P, at a location x gets mapped to a point P prime that is at x plus some displacement U. So this is the displacement vector U.

Our point Q, which is originally at some location x plus delta x, where delta x is the original separation between P and Q, gets mapped to a point Q prime, which is going to be equal to a whole collection of terms. It's going to be equal to x plus delta x, the original location, plus the linear variation of U with x that has to go U times x plus delta x. And if we simplify this a little bit, Q prime is going to be at a location, factoring out x plus delta x, x plus delta x times 1 plus e.

AUDIENCE: Is e represented here?

PROFESSOR: e is the linear relation between displacement U and position along the body. OK, so what has happened here? The relative change of length is going to be P prime Q prime minus PQ. And if we just substitute in our two expressions, the relative change of length is going to be equal to delta x times 1 plus e minus delta x.

This is the relative change of length. And I should label this such. That's going to be equal to delta x times 1 plus e minus delta x divided by delta x. And that simply going to be e, which is equal to delta U over delta x. And that is what we define as strain. And that is a dimensionless quantity. Because it has units of length over length.

All right, how can we patch this one-dimensional sort of behavior up to a three-dimensional situation? What we are saying in this one-dimensional case is that the displacement of a particular point P varies linearly with position in the body.

So let's generalize that into three dimensions by saying that the component of displacement U1-- and this is going to look like an algebraic identity. It's going to be the way in which U changes with x1 times the position x1. So this is saying that the displacement will vary linearly with x1. And the rate will be du1 dx1.

But also U1 is going to change with the coordinate x2. And the coefficient there will be du1 dx2. And there's going to be a third term that will be the way in which U1, the x1 component of displacement, changes with x3 times the position in the body x3.

So this is saying exactly the same thing that we did in one dimension, that displacement depends linearly with position, except that there's now three coordinates for position. And the displacement will change with an increment of a change in position along each of those three axes.

Similarly, we can say that U2 is going to be equal to the way in which U2 changes with x1 times the position x1, the way in which U2 changes with x2 times the coordinate within the body x2, and the same for U2 dx3 and the actual positional coordinate x3. And so, in general, we're going to propose that the i-th component of displacement is given by the way in which displacement changes with x sub j times x sub j.

Or you could do the same thing, not in terms of actual displacement, but differences in displacement, the same way we could define strain as the fractional change of length of our line segment on the elastic band or, alternatively, as the shift in position. So we could define it also as the difference in distances or displacements let me call them.

And that really is a trivial change. We'll say the change of a length along x1 delta U1 is going to be the way in which U1 changes with x1 times delta x1 plus the way in which U2 changes with x1 times the way in which U1 changes with x2 times delta x2 plus the way in which U1 changes with x3 times delta x3 or, in general, that the fractional change in length is going to be equal to dui dxj times delta xj.

All right, so what we're going to define now is dui. Dxj will be defined as something like strain. It's not exactly strain yet. And this is going to be a measure of deformation. I'm hedging my words because of something that we'll want to impose upon the proper strain tensor.

But first, let's see what these three-dimensional terms mean. And to see that, let me look at a place within the body. And I'm going to look at a line segment that goes between a P and Q that is oriented along x1. And I'm picking that deliberately. Because I want to keep things simple and isolate one of these terms so we can identify its meaning.

So let's say here we have two points P and Q separated by a distance delta x1 prior to deformation. And now we squish the body. And what happens is that P will move to a position P prime. Q will move to a position Q prime. And this will be the new line segment, P prime Q prime.

This will be the original delta x1. To that we're going to tack on an instrument which is delta U1. But then there will also be a delta U2. And clearly what has happened is that the length of the line segment P prime Q prime has changed. But also the orientation of the line segment has changed by an angle, which I'll define as phi.

So again, we have a line segment. We deform the body. P is displaced. Q is displaced. The component of that line segment along x1 has changed in length by an amount delta U1. The coordinate along x2 will change by an amount delta U2. So we not only change the length. But we rotate the line segment.

We can express these in terms of our general relation that I proposed a moment ago. Delta U1 is going to be equal to du1 dx1 times delta x1. And that is going to be given by the element epsilon 1, 1 times delta x1. The value of x2 was going to be equal to du2 dx1 times delta x1. And that's going to be by definition e2, 1 times delta x1.

And the reason this is so simple is that I initially picked the line segment which was parallel to x1. So not all of the four terms that would be present in the x1, x2 system have come in. So there's no contribution of delta x2 because of the fact that I've picked this special orientation.

OK, the fractional change of length resolved on x1 is going to be, well, it's going to be exactly delta x plus delta U1 quantity squared, it's going to be this horizontal line segment, plus delta U2 quantity squared all to the power 1/2.

And now I'm going to say that delta U2 is negligible compared to this big delta x plus delta U1. And I'll say that this is approximately equal to delta x plus delta U1, OK, just taking the whole works outside of the square root sign.

So delta U1 over delta x1 is going to be equal to the term e1, 1 from this expression here. So the term e1, 1 represents the tensile strain along x1. So we can see how these derivatives are going to enter into changes of length.

The P prime Q prime has also been rotated by phi. And we can say exactly what that is that the tangent of phi is going to be given exactly by delta U2 divided by the original length of the line segment delta x1 plus the little increment of displacement along x1.

And this is-- I'll walk down here so I can see it. This is delta U1. This is delta U2 over times delta x1-- sorry, I can't see what I've got down here-- delta U1 plus delta x1.

OK, so it's the amount of displacement along x2 over the original line segment delta x1 plus the change in displacement delta U1. And clearly delta U1 can be claimed to be small with respect to delta x1. So this is approximately equal to delta U2 over delta x1. And tangent of phi is going to be tangent of a very small angle. So this will be delta U2 over delta x1.

And so this angle phi is, for small strains, going to be equal to delta U2 over delta x1. And that is the definition of our element of strain e1, 2. So e1, 2 corresponds to a rotation of a line segment that was originally parallel to x1 in the direction of x2--

AUDIENCE: [INAUDIBLE].

PROFESSOR: e2, 1, I'm sorry, e2, 1, yeah, along x1 in the direction of x2. And that is counterintuitive as I have just demonstrated. e2, 1 is a rotation of a line segment along x1 in the direction of x2, which is just the reverse of the subscript.

So eij, a general off-diagonal element of the array eij, is going to be a rotation of a line segment initially along x sub j in the direction of x sub i. And for very small strains, numerically that term eij will give an angle in radiants.

AUDIENCE: For the equation you have over there, should it be delta x plus delta x2?

PROFESSOR: No, I would say this--

AUDIENCE: Because aren't you saying delta U1?

PROFESSOR: Oh, OK, you're right. Yeah, I didn't take the difference here.

AUDIENCE: Delta U1 is basically negligible?

PROFESSOR: Yep, yep, so I'm saying that that [INAUDIBLE] should be delta x1--

AUDIENCE: Plus delta U2?

PROFESSOR: --plus delta U2. You're right. No, I'm throwing this out. And that's right. So I'm saying that this is essentially delta x1 plus delta U1. I'm saying that this is negligible.

So strictly speaking the distance between P prime and Q prime is the square root of this squared plus this squared. But if this thing is tiny, P prime Q prime is essentially going to be this distance, delta x1 plus delta U1. So that's right.

OK, so we have something that looks like it measures deformation. But I would like to ask if this is a suitable measure of deformation. And I would like to show that, unless the tensor eij is symmetric, that we have included in our definition of strain rigid body rotation as well as true deformation.

So in order to demonstrate that, what I'm going to look at is a case where we have actually, by the way in which we apply a stress to the material, actually done nothing more than rotate x1 to x1 prime and rotate x2 to x2 prime. And in general, for a real amount of deformation, that is something that is going to happen.

Let's say I decide to deform this eraser in shear before your very eyes. And I try shearing it. Nothing much has happened. And I squeeze a little harder. And finally, I've got it wrestled down into an orientation like this.

And when I finished, this was the original position of the eraser. Here's x1. Here's x2. And by the time I've wrestled it around to a deformed state, it sits down like this, maybe deformed a little bit. But would you say that this angle here is a measure of deformation? No, that's clearly a rigid body rotation.

And what I would intuitively do, if I wanted to measure true deformation, if this were the original body and after deformation it went to a location with some deformation to be sure but this has been rotated to here. This has been rotated down. And I wouldn't want to say that this is a measure of deformation. This would be e1, 2. What I would want to do intuitively would be to position the body symmetrically between the axes and then say that this is a measure of shear strain.

So let me show you now in a more rigorous treatment that, if there is a component of rigid body rotation, let me show you what a rotation of the coordinate system to a new orientation x1, x prime would do. So let's say that this is an original point Q1. And after deformation, it moves by a rotation phi that's equal to e2, 1 to a new location Q1 prime.

Let's suppose that this is a position Q2. And after what we think is the deformation, this moves to a location Q2 prime. And this would be e1, 2. That's equal to minus phi. This is e2, 1. And that's equal to plus phi.

So what we would do is like to get rid of that rigid body rotation. And what I'll do is to show that, if we have a point that's out at the end of a radial vector, R, which has coordinates xi, x1, x2, x3, and if we have a displacement, which is absolutely perpendicular to that radial vector, that this would be characteristic of rigid body rotation. And I will say that, if Ui is perpendicular to the radial vector xi, then U.R should be equal to 0 for every position xi.

So if we just multiply this out, this is saying that Ui Ri should be equal to 0 for rigid body rotation. And let's simply carry out this expansion. And I will have then eij xi xj forming this dot product. And that is going to be 0 for a rigid body rotation.

And if I expand this, this will contain terms like e1, 1 times x1 x1 plus e2, 2 times x2 x2 plus e3, 3 times x3 squared. And then they'll be cross-terms e1, 3 plus e3, 1 times x1 x3 plus e1, 2 plus e2, 1 times x1 x2 plus e2, 3 plus e3, 2 times x2, x3.

And my claim now is that, if this represents rigid body rotation, then that should be 0. That is to say the displacement is absolutely perpendicular to the radius vector for small displacements. This must be 0 for all xi.

And the only way that's going to be possible for any value of x1, x2, x3 is that each of these six terms vanish individual. It's the only way I'm going to be able to get the whole thing to disappear for any value of coordinate x1, x2, x3.

So we're going to have to then have e1, 1 equals e2, 2 equals e3, 3 equals 0. All the diagonal terms are going to have to be 0. In order to get the fourth term to vanish, I'm going to have to have e1, 3 equal to the negative of e3, 1 and e1, 2 the negative of e2, 1 and e2, 3 the negative of e3, 2.

So what is this going to look like for the part of the e tensor that corresponds to rigid body rotation? The diagonal terms are all going to be 0. And the off-diagonal terms are going to be the negative of one another. So this is e1, 2. e2, 1 would have to be equal to minus e1, 2. e3, 1 would have to be equal to the negative of e1, 3. And so we will have something like this.

So this suggests that our definition of the true deformation, which will be a tensor epsilon ij-- I bet you kind of guessed I was going to call it epsilon and not E. I can write this as a sum of two parts. I'll turn that around and say I'll write epsilon ij plus another three by three array omega ij is equal to the tensor eij. That's a novel idea. This is addition of two tensors element by element.

And if I do that and define epsilon ij equal to 1/2 of eij plus eji, so for the diagonal elements that is going to take 1/2 of e1, 1 plus e1, 1. And that's just going to give me e 1, 1 back again. And I'm going to define the terms omega ij as 1/2 of eij minus eji. And if I do that, the resulting tensor will be symmetric.

OK, so from tensor eij we can split it up into two parts and a part omega ij. And I won't bother to write it out. But you can see that omega ij plus eij is going to be equal to eij minus 1/2 of 2eij minus 1/2 of 0. And that's just going to be eij. So this plus this is indeed going to give me the tensor eij.

So given a tensor eij, which is not symmetric, I will define eij as the sum of two parts, a tensor epsilon ij, which is true strain, and a part omega ij, which is rigid body rotation. And now at last we have a satisfactory measure of true deformation in terms of the displacement of points within a deformed body where that displacement can arise from either rigid body rotation and or deformation.

So finally, we have a tensor epsilon ij, which is the strain tensor. And now I can finally make my claim that, if this really is true deformation, that for cubic crystals-- you can see the same [INAUDIBLE] window coming again-- since second ranked tensors have to be symmetric for cubic crystals, the form of strain for a cubic crystal can only be this. It has to be diagonal if the tensor is to conform to cubic symmetry.

And you say, ah, ah, you tried to pull that on us with stress, too. And I can squish a crystal anyway I want. But you can only develop a strain if you deform a crystal. And the symmetry of the crystal is going to determine how the crystal deforms.

So therefore, a cubic crystal should be able to deform only in a way that stays invariant to the transformations of a cubic symmetry, right? So cubic crystals can only deform isotropically.

So if I wanted to design springs for an automobile so that elastic energy could be stored most efficiently in a solid, I would want to make these automobile springs out of a triclinic metal. Because then the tensor could be one of general deformation.

OK, well, this, obviously, is a swindle like my assertion that stress could only be an isotropic compressional stress for a cubic crystal. But the argument is a little more convoluted. And strain, indeed, is also a field tensor.

Because, even though I can only create the deformation by application of a stress or perhaps an electric field or something of that sort in that the link between the deformation and what I do to the crystal is coupled by a property of the crystal, which has to conform to the symmetry of the crystal, nevertheless, given a tensor that describes deformation, I can always, independent of the symmetry of the crystal, devise a particular set of stresses that would produce any state of strain I want.

I would have to design, though, a particular state of stress to produce a desired state of strain. And that coupling has to conform to the symmetry of the crystal. But there's no reason why strain itself has to conform to the symmetry of the crystal. I can create any state of strain I like by choosing the appropriate stress.

OK, so that is a swindle. But everything now that I can say about the behavior of a second ranked tensor applies to the strain tensor. I can take a strain tensor epsilon ij. And I can perform the surface epsilon ij xi xj equals 1. And that will be the strain quadric.

The strain quadric will have the property like any quadric constructed from a tensor that the value of the radius in a particular direction is going to be such that the strain in that direction is equal to 1 over the radius squared. So what does that mean? Well, we have to look at what the strain tensor is relating.

The direction, remember now that the definition of the strain tensor is that U sub i equals epsilon ij times x sub j. So the quote "applied vector" is the direction in a solid. And we will get, in general, a displacement, U, which is not parallel to the direction that we're considering.

And the epsilon in a particular direction is going to be equal to the part of U that's parallel to the direction of interest divided by distance in that direction. And what this is going to give us is the tensile component of deformation in the direction that we're examining.

Now, the radius normal property works in this case. And what the radius normal property says, if you look at a certain direction in the solid and then look at the normal to the surface at that particular point, this will be the direction of what happened. So I've not drawn this correctly. This should be the direction of U. And this is the radio vector.

So the normal to the surface gives us the direction of the displacement that's going to occur. And the reason I can say that is by definition that that property holds only for a symmetric tensor. And by definition, we have defined the strained tensor such that it is symmetric. So the radius normal property holds.

If we want to change from one coordinate system to another, we do that by the all for transformation of a second ranked tensor that, if we change coordinate system, the elements of strain change to new values epsilon ij prime that are given by cil cjm times epsilon lm, the same law for transformation of a second ranked tensor that we have seen earlier.

Finally, we can, because there is a quadric, we can change that by finding eigenvectors and eigenvalues, take a general form of the tensor epsilon 1, 1, epsilon 1, 2, epsilon 1, 3, and so on, and by solving the eigenvalue problem, convert this to a value epsilon 1, 1 prime 0, 0, 0, epsilon 2, 2 prime 0, 0, 0, epsilon 3, 3 prime to find out what the extreme values of tensile deformation are and the orientations within the original description of the body in which these extreme values occur.

OK, I'm just about out of time. I'll save until next time another neat feature of the strain tensor. And that is contained within these elements is information on the volume change that the body undergoes. And this is something called the dilation. And we'll see that this is related in a very simple way to the elements of the strain tensor.

OK, we've spent a lot of time developing the formalism of stress and strain. And next thing we'll do is to move on to some interesting properties of crystals, which are defined in terms of tensors of rank higher than two. So we'll look at some third ranked tensor properties. That will include things like piezoelectricity, the stress optical effect, and things of that sort.

These are going to be really anisotropic. Second ranked tensors were anisotropic enough. But the variation of property with direction was a quasi-ellipsoidal variation When as 1 over the square of the radius of the quadric. For higher ranked tensor properties, we will encounter some absolutely weird surfaces with dimples and lumps and lobes, not this smooth quasi-ellipsoidal variation, a property with direction, which was the case for second ranked tensor properties.

So we will get into some exotic anisotropies very shortly to finish up the semester. So I will probably see some of you at the MRS meeting next week. I won't see you on Thursday for reasons that I need not elaborate upon. So I hope you have a happy holiday. Relax, suck in air, and come back refreshed next Tuesday for some really wild anisotropy of physical properties.