PROFESSOR: All right. The quiz on Thursday will cover up through piezoelectricity. You've had a set of notes in your hands that cover just about everything that I wanted to say about piezoelectricity. There are other modulae that one could talk about. But these follow quite directly from the one or two that we will do.

So we will not have anything to say about elasticity until the last lecture of the term, which is a nice outcome because you certainly don't want to take a quiz on forthright tensors. You'll spend the entire hour just writing out all these cumbersome equations. All right. So the quiz will cover up through piezoelectricity, including a few of the representation surfaces that we'll examine today.

The thing that we'll be looking at is-- I'm sorry. You have a question?

AUDIENCE: Yes. A question. Well, you said [INAUDIBLE] example.

PROFESSOR: I'm sorry. Third-rank tensors.

AUDIENCE: OK.

PROFESSOR: Also, we had just a little bit of that going into the second quiz. And we didn't really ask anything about that on the second quiz. So it'd be third-rank tensors. But you have to use second-rank tensors to define third-rank tensors. So it really will be not the emphasis, but certainly you should be familiar with the early part of what we did with second-rank tensors.

All right. So today we'll look at some representation surfaces to your wonder and delight at how incredibly anisotropic the variation of these third-rank properties are with direction. One of the things that I love to do for problems when we get to these different piezoelectric effects is make up hypothetical devices just for fun.

So among those that I've invented, the first one is a earthquake sensing device because it's well known that California is going to split in half, and half will fall into the sea any day now. So if this is the San Andreas fault, I have developed large, prismatic monoclinic crystals that I embed into the San Andreas fault at regular intervals. The bottom of these crystals is grounded. There's an electrode at the top.

And if the San Andreas fault starts to move, and there is shear on these crystals, there will be a charge developed on the top. And I ask you to relate the charge on the top in terms of the piezoelectric modulae to the shear along the San Andreas fault. So there's a very clever little device that clearly is going to be lucrative because these crystals will have to be huge in size. And they'll cost more even than silicon.

Another device that I've invented is used down at the Boston fish pier. This is a crystal on which I hang a pan, and you put fish in it. That creates a tensile stress in this direction. We measure the charge on this face. And we put a little meter that measures the charge that's accumulated. So this is how the fishermen can weigh their fish after they haul them in off the boat.

So you see this is really practical material we're dealing with. This has applications in all realms of life. So today I'd like to show you, also, another problem that sets up. And I pass this out just so you can, again, have a look at some of the questions that you should be equipped to answer.

So here-- not fully expecting anybody to do it, but you can see the sorts of problems one might ask. Here is problem set number 16, which asks you, should you be so inclined, to think about manipulations of third-rank tensors. But then the second question is an idea for a device which the Center for Material Science and Engineering is considering employing here in building 13. And you can read all about that.. This is the famous soup cell.

I think probably you will see some sort of for fun modulus for a particular device on the quiz. I haven't made one up yet. But I think you'll have a look at something like that. And when we do the longitudinal piezoelectric modulus for quartz, which we'll do momentarily, this will give you an idea of how to set up these expressions.

The problem, basically, is that the direct piezoelectric effect measures a polarization in terms of all of the elements of applied stress. So there is a modulus dijk times all of the elements of stress, sigma jk. So this, until you use the condensation of subscripts, contains nine terms going this way and three equations going this way.

As we discussed earlier, since only six of the nine elements of stress are independent, you can condense this down into a 3 by 6 array of terms. The problem in doing that, though, even though one has only 18 different modulae to work with, instead of 27-- that's a considerable economy in notation, and an elimination of a great deal of redundancy-- the matrix form of this relation-- and I emphasize that it is a matrix, and it's no longer a tensor-- the matrix form cannot be transformed. Again, there are only six terms in the matrix elements of stress. And there are three equations, again, one for each component of polarization.

So instead of having 27, one has only 18. But if you are considering changing the reference axes-- and that is one of the things that it's interesting to do for these various modulae that one can define-- change the orientation of a particular rod-shaped specimen that you cut out of a crystal to different crystallographic orientations and then ask how the scalar modulus changes as you change the direction in which you've sliced out the wafer or the rod of material. In order to do that, you want to transform the piezoelectric tensor to a new set of reference axes. And you cannot transform the dij's because they are a matrix and not a tensor. And no law of transformation is defined.

So what you have to do in any generic problem of this sort is, for a particular single crystal, look up the form of the dij matrix that will have the equalities between tensor elements written in and the 0's, those modulae which are identically 0, entered into the array. And then you have to work your way backwards to get to the full tensor notation. So we'll see this when we look at the modulae for symmetry three two.

You'll have to write in the exact matrix subscripts without absorbing the equalities in the notation. And then you'll have to expand the matrix terms into full three-subscript tensor terms. Then if you want to transform the axes, which is to say you want to cut out your specimen, be it a plate or a rod, in a different orientation, you have to transform the full three subscript tensor elements to a new setting. And then if you want to continue to work in that setting in the compact matrix form, collapse it back down to matrix form, insert the equalities, and then you're back to where you started from. So this problem of seeing how modulae that describe different phenomena vary with crystal symmetry, you have to go through this problem of expanding the compact form and then collapsing back down when you've got it as a function of some orientational angle or in terms of a coordinate system that you want to work in.

The problem is exactly the same for elasticity. And we'll look at some of these modulae next term. You've heard these names before, I'm sure-- Young's modulus, shear modulus, and so on. We'll take a look on next Tuesday, a week from today, at Young's modulus, which is one of the more important ones.

And probably are used to seeing this in the form of information for a polycrystalline material, which is essentially isotropic. The modulus is much more interesting for single crystal materials. And then the surfaces that are defined particularly for the lower symmetries are absolutely wild things with lumps and wiggles and lobes and things of that sort, nothing like the dumb old, uninteresting ellipsoids that we encountered for second-rank properties.

So let's, then, take a look at how we had set up and defined the direct piezoelectric effect. We set this up as a proper tensor relation, saying that P1 is equal to d1. V1 And then, you'll recall, we have nine elements of strain-- sigma 11, sigma 12, sigma 13, sigma 21, sigma 22, sigma 23, sigm a 31, sigma 32, and sigma 33.

But the tensor is symmetric, so we really only need to enter into our relation six of these nine terms explicitly. And what we did was to replace the two subscripts on the elements of strain, which, again, you need if you want to refer to those elements of stress through a different coordinate system. You have to know the elements of stress in tensor form.

But we convert it to six terms by going and replacing the pairs of subscripts with a single one, two, and three, marching down the diagonal of the tensor this way and then marching up the right-hand side, calling two, three, four, and calling one, three, five, and finally ending up in this slot here. And we call that six. So that was the notation we used to get to a matrix representation.

But the place where all this started is-- and I'll write just one line of this to be merciful-- d 111 times sigma 11, so this pair of subscripts goes with this pair, plus d 122. I'm putting that one in next because we're going to number the terms for stress in this order, one through six. Times sigma 22 plus d 133 times sigma 33 plus d 123 times sigma 23 plus d 132 times sigma 32 plus d113 times sigma 23 plus d 132 times sigma 32.

And finally we end up in slot number six. And we have a d 112 times sigma 12 plus a d 121 times sigma 21. Look at that. There's an equation that covers two whole blackboards.

So if we now condense this down to matrix form we would say that P1 is d 11 times sigma 1 plus d 12 times sigma 2 times d 13 times sigma 3 plus-- and now we have this messy problem with the 2's-- we have a d 14 times sigma 4 plus, again, a d 14 times a sigma -- 2 3 is equal to 32, so I can call this 4-- and then these terms become 15 sigma 5 and, again, a 15 times sigma 5 plus a 16 times sigma 6 plus d 16 times sigma 6.

Now we have to make a choice. Either we are going to have, in a general matrix relation, that P sub i is equal to dij times sigma j, if j is equal to 1, 2, or 3. But it's equal to 2 dij times sigma j if j is equal to 4, 5, or 6.

And that is something we like to avoid, if possible. That's ugly. That's ugly. And it's going to be a hell of a matrix if we have factors of two in front of some of the matrix elements but not in terms of others.

So this is something we could do. Hey, it's our ballgame. We make up the rules. But that's going to be an ugly thing to have to deal with.

So instead, as we mentioned last time, what we will do is to lump together these terms and define d 14 not as these individual tensor elements, but define those matrix elements as the sum of these two elements. And then we saw before that-- we saw last time in our earlier meeting-- that from the converse piezoelectric effect, which expresses strain in terms of an applied field, where the elements of strain 1, 23, and 32 appear in separate equations. And knowing that the same array of piezoelectric coefficients amazingly describes the converse piezoelectric effect as well as the direct piezoelectric effect, we know that d 123 equals d 132. And that is from the converse effect.

So equivalent to saying this is to define d 14 as twice d 123 because these two elements are equal. So we're eating the factor of two here so that we can write a nice matrix relation that doesn't involve a factor of two. So making this combination of terms for the shear stresses, we would have simply d 14 sigma 4 plus d 15 times sigma 5 plus d 16 times sigma 6. And we can say, in general, for the other two equations, by analogy to this one, that P sub i is dij times sigma j-- a nice, neat matrix relation but one for which, unfortunately, there's no law of transformation for the matrix modulae dij. If you want to change to another coordinate system, we have to be prepared to resurrect this full three subscript notation on the piezoelectric modulae.

OK. Comments or questions at this point? All right.

If not, let me remind you that in the notes which I distributed last time, there are summarized all of the constraints imposed on the piezoelectric modulae for single crystals. And, again, these constraints, these requirements that the tensors remain invariant for the change of axes produced by a symmetry element that the crystal possesses, these transformations show that no third-rank tensor property can exist in a crystal that has inversion. So the 11 [INAUDIBLE] group, so-called, that possess inversion have absolutely no property and can be not considered further for third-rank properties. And then one must consider all of the 32 minus 11 21 point groups that lack conversion separately. There's no reason why they should behave the same way.

Remember that for second-rank tensor properties we pulled the argument that inversion imposes no restrictions or constraints whatsoever on second-rank properties so, therefore, two different symmetries that differ only by the presence or absence of an inversion center. That is to say, you change 2 to 2 over m if you add inversion. But the argument was inversion requires nothing, so the constraints or symmetry, too, look exactly like those for symmetry. And here you've got to plod through every single one of the non-centrosymmetric point groups separately. And they all have tensors that have different forms.

So what I'd like to do is look at one specific one. And that is the matrix for symmetry 32. And that is a point group that you'll recall has a threefold axis and twofold axes at intervals of 60 degrees. And in your list of the qualities and absences, 32, where the 3 is parallel to the axis x3, and the twofold axis is parallel to x1. So we're defining this as x1, this as x3, and x2 comes out halfway between the two.

So let me draw this looking down along the threefold axis. These are all twofold axes. And we'll take x1 in this direction. x2 pokes out in between two twofold axes. And x3 comes straight up along the threefold axis.

With that coordinate system, the form of the piezoelectric modulus matrix has this form-- d 11 minus d 11 0 d 14 0 0 0 0 0 d 15 minus d 14 0 and in the bottom row d 31 d 31 d 33 0 0 0. So this matrix with the equalities put in is obtained by looking at the full three-subscript tensor and requiring that it look the same before and after any of the rotations involved by these three distinct axes-- the threefold and the pair of twofolds.

OK. Now, there are many different scalar modulae that one could define, some of them serious and worth the consideration because of the application in devices or other practical situations. The modulus that I'd like to examine is something called the longitudinal piezoelectric effect.

And let's emphasize, again, that it is impossible to come up with one representation surface that fits every need because, again, the direct piezoelectric effect relates the components of a vector to a tensor, sigma ij. There are six independent tensor elements. So how can you describe how this vector is going to change as you change orientation of a crystal whose behavior is described by all of these modulae?

But I would point out, however, that there are only two distinct-- oops, I'm sorry. This is d 14. I don't know how I made that d 15.

There are only, in this array-- and I slipped a notch. Excuse me. Last couple of days are such that I am not able to even read from my notes.

So these bottom lines, my apologies, are all 0. And there are two modulae, d 11 and d 14. So there are two independent numbers, but they appear as different matrix elements and, therefore, different tensor elements, as well. And this should be minus 2 d 14.

I'm sorry. I slipped down a notch, and I got some for 32 and some for 6. And I'm glad I found it at this point, or I'd really be in deep trouble. OK. So two numbers and that is, in fact, the form of the matrix for symmetry 32.

So the longitudinal piezoelectric effect is one of the representation surfaces that gives you the way in which the polarization will change for one very, very specific type of stress. In particular, what we'll do is set this up as a coordinate system. And we will look at a coordinate system where this is the reference axis, x1. We'll come down with a compressive stress, sigma 11, along that axis. And, therefore, we're applying a uni-axial stress, which has just one component of stress.

So that's very specialized. In general, there would be six different components of stress. But we're looking at one specific stimulus applied to this crystal plate.

In response to that sigma 11 there are charges induced on all of these surfaces. And these charges, this charge per unit area, is proportional to the component of polarization P1, the component of polarization P2 along the surface out of which x 2 comes, and the charge per unit area or the polarization along x3. So this would be P3. And this would be the direction of x3.

Now I deliberately tried to show this sample as a thin wafer, which has a much larger surface area here than it does on the other two surfaces-- a much smaller area there. Therefore, since polarization is charge per unit area, if this area normal to x1 is a very large area, there's a lot of charge accumulated there. It's going to be easy to measure. If we make the wafer vanishingly thin, then the charge per unit area is high, but the total area is small. So there's going to be a negligible accumulation of charge on these two side surfaces.

So the longitudinal piezoelectric effect and the longitudinal piezoelectric electric modulus is an effect, where we look at the component of polarization, P1, in response to an applied stress, sigma 11. So it's that simple. Look at all the terms we've thrown out. We've thrown out a whole bunch of elements of stress, which we could impose if we wanted to. And we've thrown away two of the three components of polarization by designing a specialized sample.

So all that's left then is that P1 equals sigma 11. And the relation between those two parameters is the 111. So all this is going to hinge on one single piezoelectric electric modulus, d 111, and how that changes with direction.

So this is for one orientation of a plate. And I had not specified how the orientation of this plate is related to the symmetry axes. So let's do that now.

What I'm going to assume is that this is a crystal. It doesn't look like it's hexagonal. But imagine that this is a crystal of quartz.

And we could look at an x1 that's in this direction. And imagine that we have cut out of this crystal a wafer that has a normal along x1. And then relative to this coordinate system, if this is x1, the modulus d 111 would tell us what charges accumulated on these two surfaces.

But now, what we could do if we wanted to know how this modulus changed with direction would be to cut out a plate, a thin plate, in another orientation, where this is x1 prime. And this has changed relative to the orientation of the cell edges in the crystal. The crystal is fixed. We're just cutting a wafer out in a different orientation. So this is x1 prime.

We're going to, again, squeeze it with a tensile stress sigma 11 prime. And we'll ask how the polarization P1 prime is related to sigma 11 prime. And the answer is that P1 prime will be a tensor element d 11 prime times sigma 11 prime.

So what we are asking, essentially, is how does d 11 transform when we take the direction of x1 in a different orientation and, thus, change the value of d 11 prime? It's going to change all of the piezoelectric matrix elements. But we're looking at an effect in a sample that is deliberately prepared such that we will measure only the surface charge given by P1 prime. And, therefore, the way in which the properties of this plate change as we vary the way in which we've sliced it out of the single crystal is going to be simply the variation of d 11 prime with direction. So this is the general nature of what we will do when we define any of the scalar modulae related to the piezoelectric modulus tensor.

We can change our notation a little bit in that we have a modulus which I'll define as a scalar modules d. And that d is going to be d 111 prime. And I know how to evaluate that. d 111 priime will be C 1l, C1m, C 1n, where these are direction cosines, times all of the elements in the original tensors, dlmn, in the tensor referred to the original coordinate system. So even though this looks simple-- it's just one modulus-- when we transform it, we've got a product of three direction cosines out in front at every single one of the 27 tensor elements in the original tensor. So it's not as trivial as it seems.

So this is how this modulus that relates compressive stress to induced surface charge will change with orientation. But what are these direction cosines? These are the direction cosines-- not the full direction cosine matrix. These are the direction cosines for x1 prime. OK?

So we can get rid of this two-subscript notation if it's understood that these are the direction cosines of x1 and simply call these l l, l m, and l n, just as we did for the direction cosines of a vector because we're only concerned about the orientation of one of the axes, namely x1 prime. We don't care diddly-bop about x2 prime or x3 prime because these don't enter into the modulus that we have defined. And this will be times dlmn.

OK. Is what we're doing clear?

So we have defined this particular effect in terms of those of the 27 piezoelectric modulae which are necessary to describe it. And then we've established how they will change with a change of the direction of one particular direction. And we don't care anything about x2 or x3.

All right. Now we go through this process of inserting for matrix notation with the equalities built in. The proper matrix notation in the first term is d 11 in matrix notation. That's this term up here in the upper left-hand corner.

The second term, the term that we've written as minus d 11, that's not d 11 at all. This is, by definition, d 12. And we need the true subscripts, if we're going to transform this. So this really is not even a matrix because the subscripts have lost meaning, and we're just using them to identify equalities.

Then comes a 0. And next comes something that we've labeled d 14. And the subscripts there are correct.

That is, indeed, the fourth term in the first row. But we're going to want to convert d 14 into a tensor element momentarily. Now let's get the rest of the terms that are non-zero.

This is really d 25. So the next term that is non-zero is d 25, which just happens, because of symmetry, to be equal to minus d 14. But this is the true matrix subscripts, and this is the true matrix subscript here.

The next term over to the right is the fifth and final non-zero term. This is minus 2 d 11. And this is really d 26. Those are the true matrix elements.

So we put in the proper matrix subscripts. And now the next, final, step in the expansion is to convert these terms into actual tensor elements. So this is d 111. And these are tensor subscripts, so this is something we can transform.

This is d 12. In tensor notation this is d 122. And that's something that has a law of transformation. d 14 is really d 123 plus d 132. We lumped two tensor elements together to define this modulus.

Minus d 14 that appears in the next to the last non-zero spot is d 25. d 25 is really d 231 plus d 213. Then, finally, d 26 is d 121 plus d 112.

AUDIENCE: Shouldn't that be d221?

PROFESSOR: Sorry. d 26, you're right. That's down in the second row. d 221 and d 212.

Now we've got something we can transform. The law for transformation is l sub l, l sub m, l sub n, dlmn. So And these are the direction cosines of x1.

So this term will transform as l 1, l 1, l1 times d 111 . 1 The next term will transform as l 1, l 2, l 2 times d 122. And that will be d 122 prime for different orientation of x1.

This will be two terms. This will be l 123. And I can write them in any order.

So this is l 123 times d 123 plus d 132. And these terms prime, when we change axes, are going to be equal to l 2, l 1, l 3 times d 231 plus d 213. And this last term will be l 1, l 2 squared times d 221 plus d 212.

All right. So this now is our new tensor element, d 11 prime. And that's given by this sum of terms. So we'll have a first term l 1 cubed times d 111. And if I look through these other terms, that's the only term in l 1 cubed that I'll have.

The next term will involve the product of three cosines-- l 1 and l 2 squared times d 122. And if I go down here, here's an l 1, l 2 squared again. So I have plus d 221 plus d 212. And then, finally, the other coefficient that I have is plus l 1, l 2, l 3-- which is what this should be. And that will be times the sum of terms d 123 plus d 132.

Up here, you've got the same thing again-- plus d 231 plus d 213. And I have a total of 1, 2, 3, 4, 5-- 1, 2, 3, 4, 5 terms. OK, that is how the longitudinal piezoelectric modulus will change as we change the direction of the normal to the plate that we have cut out of the crystal. So these are direction cosines relative to the crystallographic axes.

l 3 is the angle between the normal to the plate and the threefold axis. l 1 is the angle cosine to the angle between the normal to the plate and one of the twofold axes. And l 2 is the direction cosine for the normal to the threefold axis and the twofold axis. OK. So we've got it now in terms of tensor elements. And now-- yeah?

AUDIENCE: Is it at all reasonable to assume instead of taking those sums in d 123, d 122, just saying 2 d 123? Is that OK in assuming? Or not necessarily?

PROFESSOR: Well, we could do that. But I did it the long way to not obscure what we're doing. OK? This is a well-defined summation over subscripts. And we're going to collapse immediately down to the sums. And we're going to replace the equalities.

So let's see what comes out of this, if we now, having reached the zenith, having transformed the tensor elements, go down and replace this with a consolidation of terms and an insertion of the equalities between the matrix elements. OK The first term is d 11. So I will have l 1 cubed times d 11.

Notice I'm getting third powers of direction cosines, which is going to be what causes the exotic nature of these anisotropies. And then I have a product of l 1 and l 2 squared. And this is d 12. And this second term is-- where did it go? This is d 21 plus d 212. And that is what we call d 26.

And then, finally, this product of three different direction cosines-- l 1, l 2, l 3. And we have d 231 plus d 213. And this is d 25.

And, again, an l 1, l 2, l 3 times d 14. And the second term here is d 25, if I've done it correctly. d 25 -- this is d 24. And this one is d 24. 2/4 OK.

Let's now insert the equalities-- back to where we came from. d 11 is d 11. d 12, however, for symmetry 32-- I'm going to my handy-dandy chart of symmetry restrictions. I don't want to do that. That's fourth rank.

AUDIENCE: It's still on the board.

PROFESSOR: It's still on the board? Yes. Thank you. When your nose is in it, it's hard to see.

d 12 is minus d 11. 1 And d 26 is minus 2 d 11. So these two terms can be consolidated. l 1 cubed plus l 1, l 2 squared times d 11 minus d 11. So these two terms die. And I have a minus 2 d 11 that's left.

If I insert the equalities here, I'll have l 1, l 2, l 3. d 25 is minus d 14. And here's a d 14 itself. So these two terms die.

And then I had d 25. And that is--

AUDIENCE: You don't have d25.

PROFESSOR: I don't have d 25. Where did I get the extra one?

AUDIENCE: [INAUDIBLE].

PROFESSOR: OK. I'll take your word for it. And I know how it has to turn out. OK. So these two terms kill each other.

And I'm left with, then, an l 1, l 2, l 3. Or have I left something out? This is d 1-- this is d 15 and d 231 and d 23 -- uh, this is d 2 -- and the combination of 13 and 31 is d 25. Right?

And if I look at my equalities, this is l 1, l 2, l 3 minus d 14 plus d 14 plus d 25. And d 25 is minus d 14.

AUDIENCE: Why would you add this d25?

PROFESSOR: Let me check my notes and see what I've got here.

AUDIENCE: [INAUDIBLE].

PROFESSOR: OK. See what I -- d 25 is minus d 14. And then I have just a d 14. I don't know -- I see what I did. I put it in the wrong slot.

This is d 14. And d 25 is the one that's minus d 14. so this term dies, which is nice because that cross term is messy.

So what I'm left with, then, if I check against my notes, is d l 1 cubed plus l 1, l 2 squared times d 11. And then I have minus d 1 minus-- this doesn't belong in here. This wants to end up being l 1 cubed minus 3 l 1, l 2 squared times d 11.

The first term is l 1 cubed. That's correct. The second term is l 1, l 2 squared. We've got a minus d 1 plus minus 2 d 11.

So I have minus 3. It should be a minus. Yeah, that carries down to a minus. So I have minus 3 l 1, l 2 squared all time d 11.

OK. So what we have ended up with is an expression for the longitudinal piezoelectric modulus as a function of orientation. The surprising thing is that l 3 does not appear here at all. It doesn't depend on the angle between the normal to the plate and the threefold axis. It depends only on one modulus, and that is a remarkable thing.

This says that the shape of this surface is independent, essentially, of the property, any property that relates the one one prime to a uni-axial stimulus, sigma 11. And you measure a vectory component in the same direction is always going to have this universal surface. And it involves just a geometric term and then one modulus that changes the magnitude of the longitudinal piezoelectric modulus but does not change the asymmetry.

So let's see what this function looks like as a function of direction. Maybe we better wait for that until we come back because that's going to take a few minutes. So this is what we found. That is correct.

And we have to now decide what this looks like, which will take a few more minutes. But let's stop here rather than run late. All right. Let's take our 10-minute break as usual.