PROFESSOR: I wanted to make a few comments on the third problem set, which was a funny puzzle. And it was intended to be entertaining and get you thinking about things related to symmetry in an amusing context. And there were two problems. The first problem had a platter on which there were arranged cherries with either two or three berries to a spray, and apples and pears. And they were either black or white, which didn't make then particularly appetizing. But in any case, there were three platters and the fruits marched around on the platter. And the puzzle was, what would be the fourth platter in the sequence?

This didn't involve symmetry in any sense of the word. But yet, what was going on from picture to picture was a mapping of the motifs from one location to another. Not just rotation or translation or anything nice in the crystallographic, but still they were being moved from one location to another. And just about everybody got the right answer just by deduction.

But the reason I liked that problem, and the reason I gave it to you, is that it introduced another sort of transformation. One of the pairs of fruits not only change locations, but they switched from black to white as you went from one position to the other. And that is another sort of transformation we can introduce. And this is something we could call color symmetry. And I would observe that this is something that you are already very familiar with in patterns, in particular, in the pattern that is represented by a checkerboard where you have black squares and white squares.

This location here is a bona fide four-fold axis, as is this location here. But there other locations, such as the corners of the black and white squares, where, indeed, there is a 90 degree rotation that takes this square and transforms it into this square. But in doing that transformation, you change it from a black one to a white one, and then from a white one to a black one again, back to a white one. And that is a color symmetry. You could make it red and green if you like to be a little more attractive.

Similarly, there are mirror lines like this that are true mirror lines. The pattern is left invariant. There's another locus such as this one here where when you reflect you go from a black one to white one and then back to a black one when you do the operation twice. So there are black-white mirror planes as well as regular mirror planes. There are two-fold axes but all of the two-fold axes are black-white axes, white to black, black back to white. So that's a new sort of transformation that we can develop if we wanted to, switching the color.

Does this is have any utility in the physical sciences? Be nice for wallpaper designers, but does it have utility in physical sciences? And the answer is yes. Because what is going on here is that there is a binary character assigned to each of the motifs. So any time the atom or whatever you have, the molecule, has some sort of binary characteristic that it can exist in two distinct states, the arrangement of that sort of motif requires a black-white symmetry.

And one example in crystals that requires this sort of symmetry is the case of a magnetic structure-- and I think I mentioned this last time-- where the magnetic moment can either point up or down. Well, saying that you have an atom with spin up and an atom with spin down is the same thing as saying, in terms of transformations, having a black atom or a white atom. It's a binary character that switches from atom to atom.

Could you have three color symmetries? Yeah. You can. There is a Dutch artist named Maurits Escher who spent much of his, career deducing patterns, and did this in a purely intuitive way. I mean you look at these patterns and say, wow, that guy is some crystallographer. But I heard him, while he was still alive, speak twice. And he describes it in metaphysical terms. This is the twoness merging with the fourness, and just completely intuitive. But his patterns are gorgeous. And he has a number of very nice ones that are three-color symmetries. And maybe later on in the term, I'll bring in a projector and show you examples of some of these. They're very entertaining.

All right. So the purpose of that first puzzle was to introduce you to black-white symmetries. Then the second puzzle on the sheet was a pretty dumb thing. There was a stack of blocks and you said how many-- I asked how many blocks are there in the stack? And you could count them up, 1, 2, 3, 4. And If you did it right, you came out with 32. So that was pretty dumb.

Well, this was intended as an example of how one can use symmetry in the solution of a physical problem. Let's suppose that you were a cowboy and you had to go out and buy horseshoes for a herd of horses. Would you count the number of feet? No. You'd count the number of horses and multiply by four. OK. So there you'd be using symmetry.

Now The point of this problem really was-- You could have had this one. We have just the exact number seats. The point of this problem was that you have to, in assigning a symmetry to a system, usually make physical assumptions. When you do a derivation, as we've been doing for the last several days, we say this is a four-fold access, and there's no ambiguity whatsoever. Because it's my ball game, and I want it to be a four-fold axis,

But when you're given a pattern or given a crystal and you have to assign a symmetry to that crystal, you invariably have to make assumptions. The person buying the horseshoes, for example, has to assume that five- and six-legged horses are pretty rare articles and that a horse with only three or two legs is going to be of little utility to the rancher. It Would probably have a limited existence at the ranchers expense. So you're assuming that all horse have four legs.

In looking at a crystal, though, things are not that easy to decide. And let me give you a few really practical examples. A lot of mineral crystals grow in fissures in the rock. And let's suppose we've got a crack, and bubbling through this crack is a solution that contains silica. And as the solution cools off, it will deposit a crystal of quartz. And this crystal of quartz would probably grow looking something like this. And you would knock the crystal out of the rock. And you would say, what is the symmetry of this crystal?

It doesn't appear to be anything hexagonal about it. But what's happened is this crystal if it had developed in a uniform environment would have had a hexagonal shape. But because the nutrients are flowing in from one direction, it tends to elongate in that direction. So how could you assign the symmetry to that crystal that you dug out of the rock? Unfortunately, crystals do not come with a legend on the bottom that says quartz SIO2 space group P3 subscript 121, made in USA. You have to yourself assign the symmetry.

And what you would do is you would look at this face and this face and you say, gee, these angles, when I measure them, all turn out to be 120 degrees. This face has a luster on it that looks like the luster on this face. This face has little edge pits on it, and they point in the same way as the edge pits on this face. I can measure the electrical conductivity and find that it varies in a fashion consistent with a hexagonal crystal. And you would do any physical test that you decided you would need to make. And then when you were all done, you would say everything that I can measure about this crystal other than it's shape is invariant to a 120 degree rotation.

Quartz has a hexagonal shape very often, but actually only this face, this face, and this face are symmetrically equivalent. And there's only a three-fold axis in there. And if you look very carefully at this crystal, you could see some difference there, too. The neighboring faces very often have striations, very faint striations on one face, no striations on the neighboring face. So you find out all that can. You have to make some assumptions, and then on that basis decide the symmetry.

Let me give you another example. This is a real one. And one of the things that it's fun to do is to grow crystals from solution. The Science Museum has little kits that you can buy that have some powder that you could heat up in water and then put it in a beaker and put a string in there, maybe with a little crumb of the stuff that you dissolved.

And one of the crystals that is nice for this purpose is alum. It's a cubic crystal. And if you let the stuff set there, it will form a nice, lovely equiaxed cubed. Science Museum usually puts a pinch of another salt in there so the crystal is not only pretty and shiny, it's purple or it's the orange.

Now, unless you attach this little seed crystal carefully, there would be a good chance that it would fall off and land on the bottom of the beaker. When that happened, your crystal would not have the shape of a cube. It would have a shape in which these two top edges had length L and these faces on the side were exactly half that length.

And what's happening is that this crystal is growing in an unconstrained environment. The nutrient is coming on and crystallizing from all directions. In this case, nutrient can't get in from the directions that are below the crystal. So the growth in that direction is eliminated, and this height is exactly half that at the edge of the crystal. So that's sort of a very specialized case of a crystal growing in a constrained environment.

I'm digressing now, but let me make a few more remarks. The shape that a crystal has-- and the shape is in crystallography designated by a special term. It's called habit. Crystals that undergo plastic deformation never die, they just develop bad habits. Oh, come on. This is a tough crowd. I thought that was pretty funny.

Anyway. The habit that a crystal has can depend very much on impurities in the solution or melt from which it grows. And let me suppose I start with a little crystal that looks like this. And there are two different kinds of faces on there. And let's suppose one of them grows rapidly and one of them grows slowly. Which face do you think would be the one that eventually forms the boundaries to the crystal volume? The fast-growing face or the slow-growing face?

AUDIENCE: Fast-growing face.

PROFESSOR: Fast, slow are both possibilities. Let's do a time lapse experiment. Let's suppose that this is the fast-growing face. And, after a certain amount of time these faces would have all advanced to this location. And this is the slow-growing face. And after the same amount of time, it will have advanced to this location. After the next time increment, this one would have grown up to here. This would only have grown that far. And you could see the only thing that's gonna be left is the slow-growing face. So it's sort of counter intuitive. The faces that survive to bound the crystal as rational planes are the slow-growing faces, not the rapidly growing faces.

This is of great use to the people who want to control crystal shape. And this is very often in very every day, ordinary contexts. The people who grow table salt, for example, want to grow a crystal that has nice pointy edges so when it comes down on your tomato, it doesn't bounce off. It sticks to it. So the shape of table salt makes a difference in how it behaves. And the way you change the shape is to find something that will be absorbed preferentially on one set of faces that you would like to have be the faces that appear on the crystal. And that will do the job for you.

I'll close with just one final example of this in real, every day industry or technology. I once had a phone call from a company that made salt for sprinkling on roads in the winter time, particularly around New England when the roads tend to ice up. And in an earlier era it, was politically correct to use sodium chloride. But you wanted the sodium chloride, if you could, to have an acicular, a needle-like, shape to improve its flow properties so you didn't waste a lot when you shook it out of the back of the truck.

And the material that they use for that-- you think rock salt in the environment is bad-- they used a cyanide compound to modify the growth habit and make it needle-like. And they wanted to get away from that before they were really dragged into some sort of trial by the EPA. And did I know of something that could change the shape of the salt crystals for them? And the answer to that question was quite straightforward. No. That was the end of the conversation. But it was an interesting example of how surface chemistry can change crystal shape.

OK. So all that long-winded explanation was a justification of the meaning of those two final two puzzles. And the final thing I have to say about them is that these puzzles were lifted from a puzzle book published by MENSA. Have any of you heard of MENSA? MENSA is an association of self-declared geniuses. So this was a puzzle book for geniuses. And you all did horribly well on that third problem set.

Everybody at MIT is bright. But you folks are not only bright, you are geniuses because you cracked the MENSA puzzles.

OK. Enough enjoyment and fun and games. Let's get serious, back down to what we were doing earlier.

We had established in earlier discussion that there are a very limited number of ways in which we can arrange a mirror plane and a rotation axis about one fixed point in space. And these, accordingly, are called the point groups. And in particular, they are the two-dimensional crystallographic point groups.

We had rotation axes by themselves, 1, 2, 3, 4, and 6. And these are the only ones we need worry about because these are the only rotational symmetries that are compatible with translation. Then we had a mirror plane. And then there was no reason why we could not combine a mirror plane with these rotational symmetries. And in doing so, we used a theorem that said if you take two reflection operations, sigma 1 and sigma 2-- and again, I use sigma to represent an individual operation of reflection-- and a mirror plane is the locus of operations of reflection, and then reflecting back again, there are two operations involved.

And if these are combined at some angle mu, sigma 1 followed by sigma 2, when they are separated by an angle mu is the same as the net transformation of a rotation about their point of intersection through twice the angle which the mirror planes are combined. And that led us-- in addition to a one-fold axis, a two-fold access, a three-fold axis, a four-fold axis, and a six-fold axis-- let us combine reflection with these axes to get symmetry combinations that we named according to the symmetry elements which were present in the final combination.

We used two mirror planes if the mirror planes that arose where independent. Independent in the sense that the two-fold axis never turned one into another. And independent in the sense that if we draw a representative motif, the way those motifs were disposed about one mirror plane was different from the way they were arranged about the other one.

For a four-fold axis, we could add mirror planes. And they would have to be at half the angle of the four-fold axis. And as we saw, it was kind of a mouthful to call this 4MMMMMMMM. There were just two kinds of mirror planes, so this one was called 4MM. 6MM, analogously, was a six-fold axis with mirror planes separated by 30 degrees, half the throw of the six-fold axis. And the one that I left out is 3, not m m but 3M because a three-fold axis has mirror planes separated by 60 degrees.

And the same thing goes on at each of these mirror planes, pair of objects hanging at one end of the mirror plane, the other end is unadorned. So they all behave the same. There's only one kind of mirror plane. The three-fold axis maps one mirror plane into another, and so they have to be the same. So therefore, only one M appears in the symbol.

Independent of this, we had, when we combined reflection or rotation with a lattice, found that there were a very limited number of lattices in two dimensions, the oblique lattice which had T1 not equal to T2 and T1 inclined to T2 by some general angle. The next degree of specialization was for a two-fold axis-- so either a one-fold or a two-fold axis could fit in here.

For a three-fold axis or a six-fold axis, we had to have a lattice in which the two translations were identical to one another, identical in magnitude and identical in the way the atoms were ranged relative to these translations. And the angle between T1 over T2 had to be identically 120 degrees. And by convention, we take the larger of two angles, 120 rather than 60, to define the inter axial angle.

A four-fold axis required a net that was exactly square, not approximately square but exactly square. So T1 and T2 had to be identical in magnitude. The angle between them had to be exactly 90 degrees. And close is no cigar. It has to be exactly 90 degrees because that angle is generated by symmetry.

And then at the place in which I'm gonna pick up today in just a moment is the case of a mirror plane and reflection, we saw depending on how we arranged the translations relative to the reflection plane, could give us either a lattice in the shape of a rectangle-- and here T1 and T2 made an angle of exactly 90 degrees. But the magnitude of T1 could be anything it liked relative to the magnitude of T2, and then a double cell, which had the shape of a diamond. But it was operationally much to our advantage to pick a double cell which had a right angle in it.

And the reason for that was that it is a nightmare to do calculations of inter-atomic distances and angles or angles between faces in an oblique system. The right angle makes these calculations very simple. And the fact that this cell tells you twice as much of the area as you really need to know about is a small price to pay. So this is the rectangular net, as we'll call it in words. And this is one that we'll call the centered rectangular net.

OK. So let me pause here and suck in air and see if there are any questions. This is where we left off. And now having everything spread out on the board, we're going to start to make combinations and continue that process. Yes, sir.

AUDIENCE: So the centered rectangular has the same exact [? intercept ?] as the rectangular but the only thing you're doing is you're deriving it based on symmetry that you're using like a diamond kind of thing to get the center one?

PROFESSOR: Yeah. So all these arose when we did it slowly and systematically. We said let's let this be the location of the mirror plane. What happens if we combine it with a translation? We can take one lattice point on the mirror plane since there's no unique lattice point. And then the mirror plane is gonna reflect this over to here. So here's a first translation. I've got a second non-colinear translation. Wham-o. Instant lattice.

So what I'm doing is taking this diamond shape and I am taking one translation here-- let's put some labels on this. Let's call this T1 and this T2. So this translation here, call it T1 prime, is my original translation T1 plus T2. In this translation, T2 is the negative of the-- Yeah. It's gonna be minus T1 plus T2. So I've taken linear combinations of the two vectors of the diamond-shaped cell.

And again, this is contrary to the rules we have that say pick the shortest two translations. Why? The final larger area of volume then you need. And the answer to that is occasionally the provision of a coordinate system, which is far easier to work in, is a small price to pay for that redundancy. OK.

And then the primitive rectangular net, just to remind you again, we said is there any case in which this is not true? And that case was if you take T1 exactly perpendicular to the mirror plane and then you define only a lattice row. So that gives you a second choice for T2. You can't pick it anywhere you like, otherwise it gets reflected across just as in the first case, and you have two translations that are not compatible.

And the only way that they are compatible is that if you make the translation T1 straddle the mirror plane. And that's what we've already got up here. So that's how we got the rectangular net. We could get a second distinct sort of lattice only if this translation was exactly in the plane of the mirror line. Yes, sir.

AUDIENCE: But the number and kind of symmetry elements are the same for both, right?

PROFESSOR: Exactly.

AUDIENCE: Uh, so--

PROFESSOR: So there's a curious sort of duality here. Here in the case of a hexagonal net, this is just a lattice of a very specialized shape. And does that have a six-fold axis in it? No, not unless I decide to put one in. I could put in a three-fold axis, alternatively. So here, I have one lattice that can accept two different kinds of symmetry. Here I've got the reverse situation. I have two distinct kind of lattices, both of which are happy and content with the same symmetry on them. So there's going to be a far greater number of combinations of lattice with symmetry than simply a one-to-one correspondence between lattice types and point group types.

OK. Any other questions? All right so let me now shift down to low gear and remind you of one thing that we did last time. We asked what happens if I can combine a rotation operation, a alpha, with a translation? Call this T1. Sounds like something I already did in showing that the values of alpha are restricted to values. But what I'm going to do now is use something that looks as though it's a similar starting point to arrive at a different result that we saw last time.

I'm going to deliberately look at a line that is alpha over 2 on one side of the perpendicular to T1. And then the operation of the a alpha is going to take this translation with a lattice point of necessity at the end of it, and it's going to move it over to here, alpha over 2, on the other side of the translation of the perpendicular. Then I'm going to pick this up with the translation T1 and move it and everything along it to a location like this.

And my question now is, what is a alpha followed by the translation T1? The handedness of the objects have not changed. So this has to be either translation or another rotation. And what we can very quickly show is if upon dropping a perpendicular down to the original translation, these lines are parallel. This line cuts across it. So this angle is alpha over 2. And this line is inclined to the perpendicular by alpha over 2. So this angle in here is also alpha over 2.

So the answer is that if I rotate from one location to another by rotation alpha and then pick up that motif or the entire space and translate it by the original translation T1 to get a third one-- this is number 1, right-handed, say, number 2, right-handed. This is number 3. Stays right-handed. Has to be related by a rotation.

And we next ask, what point is the point about which the rotation occurs? Then we used-- what at the time seemed like a trivial observation-- that a rotation axis in space or a rotation point in two dimensions is the locus that is left unmoved by the rotation. So if rotating from here and translating over to here is equivalent to a rotation and if the locus of the rotation has to be the point that's left on moved, bingo. That's where the rotation occurs. And it's through the same angle, so we can label this an operation b alpha.

And we know exactly where that point is gonna be. It's gonna be along the perpendicular bisector of the original translation. And it's gonna be up a distance x which is T over 2 times the cotangent of alpha over 2. So now we've got another, what I call, combination theorems. And again, this is nothing more than knowing how to complete the product of two operations in establishing the group multiplication table.

so let me write it down in the form of a combination theorem. This says that a alpha followed by a translation is another rotation-- in the same sense, both counter clockwise in this example-- another rotation b alpha in the same sense about a point that is always T over 2 times the cotangent of alpha over 2 along the perpendicular bisector. So that's how you would go about making an entry in the group multiplication table.

Sorry. I put you off.

AUDIENCE: Yeah. Can you just, like, I just want to see what you did actually to the motif in that drawing.

PROFESSOR: OK. Rather than taking an abstraction, an arbitrary line that is at alpha over 2, I put a motif in the space that doesn't necessarily hang on this particular locus that I drew for reference. And the rotation through alpha would move it to here. It's lagging behind this first line. It lagged behind the second one by the same amount. Then I pick it up and I move it by the same translation and that slides it over here, still canted over to the left.

And now what I claim is that I get from the first one to the third one-- and actually should have raised your hand on a different matter. This is out in front of the translation. So number 3 should sit here, also to the right to the translation. And the way I get from 1 to 3 in one shot is by rotating alpha b. OK.

Now, a cautionary note just so you do not use your new power recklessly. Let me emphasize that this is an escalation in operations, not in symmetry elements. And the reason for that is that a rotation axis, in general, contains a number of different rotations. An n-fold axis has n different rotation operations implied in it, including the identity operation of rotating 360 degrees.

So if I say I'm gonna drop a four-fold access in a lattice, what I'm doing is dropping into the lattice a 90 degree rotation, a 180 degree rotation, and a 270 degree rotation. And the location of the new operations depends on alpha. So these new operations are not gonna pop up at the same location. They're gonna be sprinkled over different locations.

So as an abstraction, that may be hard to appreciate. But we're going to straight away derive a couple of more additions. And then when we see what happens upon adding a rotation axis to a lattice in a couple of nontrivial cases, I'll just summarize the results and assume that you'll be able to derive them on your own.

The last time we had got around to doing just one of these combinations. And we ask what happens when you combine with a translation a rotation operation a pi. So we'll take some motif that sets up here, right-handed, rotate it by 180 degrees to get a second one, and then pick that up and translate it by T. So I'll get a third one sitting down like this of the same handedness.

And the question now is how do I get from 1 to number 3 directly? Well, let's use our theorem. Our theorem says we should go a distance x along the perpendicular bisector, which is half the magnitude of T times the cotangent of pi over 2. The cotangent of pi over 2 is zero. So we go up a distance x. That's equal to 0, all in the perpendicular bisector, which means we stay put. And-- in this class, there is always truth in advertising. Around this locus here is a rotation operation b pi, which takes the first one into the third one. OK.

And now, lickety split, I will derive again-- and I apologize for going fast because I did do it last time. Let's ask what happens when we combine a two-fold axis with a parallelogram net? So we're taking a two-fold axis plus a parallelogram net. In this two dimensional lattice there are two translations, T1 and T2, of arbitrary magnitude with some angle between them that's arbitrary. And to this lattice point, I'll add a two-fold axis. A two-fold axis involves the presence of two operations, the identity operation, and that we can forget about, and then there's the operation a pi.

By combine a pi with T1, I'll get an operation b pi that sits at the midpoint of T1. If I combine a pi with T2, we get an operation c pi at the midpoint of T2. If I combine a pi with T1 plus T2, I'll get another operation d pi at the center of the cell. And we don't have to ask what happens when we go three translations over. There's gonna be another operation of 180 degree rotation on that translation. But we really only need bother about what goes on on the edges and in the interior of one unit cell, because whatever's there has to be repeated by translation.

So, in making these additions, we need consider only independent translations-- that is, not related to one another by the rotational symmetry-- independent translations that terminate within the unit cell. And that means for the addition of a two-fold axis, I need do only what I have, in fact, just done. I want to consider a pi with T1, T2, and T1 plus T2.

If I have the operation a pi or b pi or c pi or d pi at these four locations, that is all I need have present to say that I have, in addition to the two-fold axis that I added to the lattice point, a two-fold axis halfway along T1, a two-fold access halfway along T2, and another one smack in the middle of the cell. These are lattice points. So those axes have to be repeated by translations, this one from here to here, this one from here, this one from here to here. So I'll have two-fold axes in the middle of all of the edges of the cell plus this fourth one right in the middle.

So this is an example of a two dimensional space group. It's a symmetry that acts on all of space. And the names that we will give to these combinations is a symbol for the symmetry element that we added, in this case a two-fold axis. And then we'll specify the nature of the lattice to which we've added it.

And all that I have to say is that the lattice is primitive because you're all aware at this point that a two-fold axis requires only a primitive, a parallelogram net. The only place we'll need a special symbol is when we add a mirror plane to either the rectangular net or the centered rectangular net. And then we'll have to specify which type of lattice of that shape, primitive or centered, we're dealing with. Yes, sir.

AUDIENCE: So all you did with that derivation there is prove that you don't have to worry about any rotations outside of that?

PROFESSOR: That's correct.

AUDIENCE: That's correct?

PROFESSOR: Yep. Which is very fortunate 'cause there's an infinite number lurking outside the boundaries of the cell.

I'll show you another general truth. What does a pattern look like? The pattern looks exactly like the pattern of a two-fold axis with that pattern of motifs hung at every lattice point. These new two-fold axes don't do any further repetition of the motifs. They don't do any repetition of the motif. They just express relation between things that you get when you take this pair and then repeat it by the translations.

So this two-fold access, for example, relates this to this. This two-fold axis relates this one to this one, and this one to this one, and so on. They're just relations that exist between the things that you get when you add the pattern of a two-fold axis to a lattice point.

So another generalization of what we're doing here that turns out to be valid, the pattern of motifs for a particular plane group is nothing more than the pattern of the symmetry element-- symmetry element or symmetry elements, sometimes we'll be adding more than one-- that we've added to a lattice point.

So in other words, if we take 2MM and drop that into a rectangular lattice, the pattern is simply gonna be these four atoms related by 2MM hung at every corner of the lattice. If you add it to a centered lattice, these would be hung at every corner of the lattice, and also at the centered lattice point.

So it's that simple. If you look at some of these high symmetry space groups or plane groups, wow, there's symmetry all over the place. You say, how can I draw a pattern for that? Easy. Just do what the point group does and drop it in at the lattice points. So that turns out to be a universal feature of all space groups and plane groups.

OK. Somebody had a question over here and I cut you off. Yeah.

AUDIENCE: Yeah. On that two [INAUDIBLE] diagram, wouldn't you want to use four of those because you said it was-- you don't concern ones that are-- you're only concerned independent ones, the ones you can't get by--

PROFESSOR: You're absolutely right. And let's look at this pattern again. Here are a pair that are hanging close to this two-fold axis. Here's a pair hanging close to the two-fold axis, very different arrangement, different arrangement relative to this two-fold axis, different arrangements relative to this two-fold axis. So what we're saying is-- and this is the way, in fact, we derive them-- that these four two-fold axes are all distinct. But the ones that are along the edges are just related by translation. So why do I have to put them in?

AUDIENCE: You just put them in to just show the symmetry?

PROFESSOR: Yeah. Exactly. Exactly. 'Cause if I say this is a lattice and I have two-fold axes here, that's an incomplete representation because if these are translations, I jolly well better have the same thing at all the lattice points. I'm just emphasizing the property of a lattice which is understood to be present in all of these combinations. OK. Other questions?

And I would remind you that a reasonable question or request would be, could you do that all again, please, at half speed? And I'd be happy to comply. But really this is ground that we covered last time. And I'm just reinforcing it. So now that we've run out of time, we can go on to something new.

Let me do, in the couple minutes remaining, let me make one more addition just to show you how things change when you add a rotation axis that has more than one rotation operation. Let me ask what happens when you add a four-fold axis to a lattice. We've seen that a four-fold axis can coexist with no lattice other than 1, which is dimensionally square and which has the same thing hanging about, two translations that are exactly 90 degrees apart. So what we're doing is taking a four-fold axis and drop it in at the corner lattice point and then step back before all hell breaks looks.

Now, first thing we're gonna do is to get the four-fold axis at every corner lattice point. And then we're gonna use our theorem, taking into account that when I say I'm adding a four-fold axis, I am adding the operations a pi over 2, a pi, a3 pi over 2. But let me call that, instead, a minus pi over 2, the same thing and a little easier to deal with. And then the operation a2 pi, which is the same as the identity operation. And that's dull and uninteresting, so we won't consider that addition.

We've already done this, a pi, in deriving P2. And that's the nice thing about this derivation. It's gonna snowball because as the symmetry gets higher, we've already done the work for the subgroup that contains the elements that are present in the higher symmetry. So we'll get an operation a pi over a pi here. We'll get an operation a pi here. And if we take the diagonal translation, T1 plus T2, I'll get the operation a pi here. Or I should probably call them a pi, b pi, and c pi to indicate that these are different two-fold axes.

OK. Let me emphasize again that this theorem that we have is a theorem in individual operations and not symmetry elements. If I say a four-fold axis is present, it means that all of these operations exist about that locus. I cannot say that a four-fold axis sits here. I cannot say a four-fold axis sits here. I'm gonna have to take each of these combinations in turn and then step back and see what operations exist about the various loci within the cell.

So the one that's really different is the combination of a translation T with the operation a pi over 2. And my theorem says if I put in a motif and rotate it by 90 degrees to here-- so this is number 1 and this is number 2-- and then translate it to a location that's related to the first by symmetry-- by translation to get number 3-- they will be related by an operation b pi over 2 that sits up along the perpendicular bisector by an amount x, which is T over 2 times the cotangent of alpha over 2, T over 2 times the cotangent of 1/2 of pi over 2, 45 degrees.

The cotangent of 45 degrees is unity because the two edges of the triangle are equal in length. So this says I should go up along the perpendicular bisector by an amount that is half the length of the cell. And that puts me right in the middle of the cell. So the operation b pi over 2 sits not at the edge of the cell, but at the center of the cell. And that, I think you will agree, even in this hastily sketched diagram, is the way number 1 is related directly to number 3 in one shot.

Now I could do it on the cheap from here. But let's show that we have first an operation a pi over 2. We've already got the operation c pi. Let's now combine the operation a minus pi over 2. And that says that for c minus pi over 2, I would go up the perpendicular bisector a distance T over 2 times the cotangent of minus 1/2 of pi over 2. And the cotangent of a negative angle is minus that. So I go a distance minus T over 2 along the perpendicular bisector. And what that is gonna do is to bring me to the center of the cell that's directly below. And this will be the rotation c minus pi over 2. But everything has to be translation equivalent at the interval T. So I can just move this operation c minus pi over 2 up to the center of my original cell.

So now I have at the center of the cell all the operations of a four-fold axis. So a four-fold axis exists in the middle of the cell. And let me write the results in here. I've now got all of the operations of a four-fold axis sitting here. I've got all the operations of a two-fold axis sitting here and here. So this is the final result, if I translate the two-fold axes over to the other edges of the cell.

AUDIENCE: What is that angle [? by the d project? ?] That should be 90 degrees?

PROFESSOR: That should be 90 degrees, if I drew it carefully. Yeah. So let me give the final result with a pattern in it. These are the two translations, equal in length, four-fold here, four-fold here, four-fold here. They are all the same four-fold axis. Four-fold axis in the middle of the cell. Two-fold axis in the middle of the cell edges.

And the pattern of objects is the set of things that are rotated by 90 degrees that form a square about the corner of the cell. Another one here. Another one that sits up here. And another one that sits like this. Four of them on the corners of the square. And that's all we're going to get in this pattern.

Again, it's just a pattern of a four-fold axis that is hung at every lattice point of the square net. And these other symmetry elements that arise simply are things that relate to the squares that are hanging at every corner of the square cell. OK. So the square in the center of the cell, for example, relates these four. And you can pick any other four and they'd be related as well.

The two-fold axis relates this to this. This two-fold axis relates this to this and this to this. So the pattern is just the square produced by four stamped out at every corner of the square net. And it's that simple.

The name of this thing-- what have we done? We've taken a four-fold axis, we've put it in a primitive net. I don't have to tell you it's squared because in your heart of hearts know that that is what a four-fold axis requires. So this describes completely the combinations that's been made.

And this is a representative pattern. A pattern of this sort is one that we've seen several times already in describing sample symmetries. That is the plane group of square floor tiles. It's the plane group of the square mesh that's in the overhead lighting fixtures. It's the pattern that is present in the tiles up above the lighting fixtures. It's a pattern that is very commonly present in square shirts, but not too often. It's kind of dull and uninteresting.

Yes. Fellow back there against the wall wearing P4, probably didn't know until now. Thank you. Actually, he's a shield. I made him come in. I called him up last night and said be sure you wear the P4 shirt. And he said, OK, I will.

All right. I'm getting silly. So that's tells me it's time to quit. So let's take our usual 10 minute break, And then we'll resume. And I think I'll go lickety split through the remaining plane groups that consists of combinations of a rotation with a lattice. OK. So do come back.