3.35 – Fracture and Fatigue
Problem Set 1 – Solutions
September 25, 2003

1. From the class notes we know that

$$\mathcal{G} = \frac{F^2}{2B}\frac{dC}{da}$$

regardless of whether the loading is fixed load or fixed displacement. We are told that a sample with a crack length $a = 25$ mm failed at a load $F = 158$ kN. We are also given that the value of the width B is 25 mm. We need to evaluate the dC/da term, and we are given values of $C(a)$ at two values near the value of the crack length at which the material failed. By definition $C = u/F$, so that

$$C(a_1) = \frac{u_1}{F_1} = \frac{0.3 \times 10^{-3}}{100 \times 10^3} = 3.0 \times 10^{-9} \text{ m/N} \quad \text{(a=24.5 mm)}$$

$$C(a_2) = \frac{u_2}{F_2} = \frac{0.3025 \times 10^{-3}}{100 \times 10^3} = 3.025 \times 10^{-9} \text{ m/N} \quad \text{(a=25.5 mm)}$$

Since $C(a)$ varies by such a small amount over this range, we may approximate

$$\frac{dC}{da} \approx \frac{\Delta C}{\Delta a} = \frac{2.5 \times 10^{-11}}{1 \times 10^{-3}} = 2.5 \times 10^{-8} \text{ 1/N}$$

So

$$\mathcal{G}_c = \frac{F^2}{2B}\frac{dC}{da} = 1.25 \times 10^4 \text{ N/m}$$

We are asked to give a value for K_{Ic}. To convert between $\mathcal{G}_c$ and K_{Ic} use the expression given in the text (plane strain case)

$$\mathcal{G}_c = \frac{1-\nu^2}{E}K_{Ic}^2$$

So

$$K_{Ic} = \sqrt{\frac{E}{1-\nu^2}\mathcal{G}_c} = 31.0 \text{ MPa m}^{1/2}$$

2. Start with the expression for the *anti-symmetric* part of the general equation for χ derived in class. Recall that in class we used the symmetric part to derive the mode I fields. So we have for this problem that

$$\chi = r^{\lambda+2}(A_2 \sin \lambda\theta + B_2 \sin(\lambda + 2)\theta)$$

Apply the same boundary conditions we used in class:

$$\sigma_{\theta\theta} = \sigma_{r\theta} = 0 \quad \text{for} \quad \theta = \pm\pi$$

These correspond to imposing the condition that there can be no stress on the crack faces. These conditions are the same for mode I, mode II or mode III. Note that no restriction is imposed on the stress σ_{rr}, since it does not contribute to the stress on the crack face.

We find that we must have

$$(A_2 + B_2) \sin \lambda\pi = 0$$
$$(\lambda A_2 + (\lambda + 2)B_2) \cos \lambda\pi = 0$$

From the same physical reasons discussed in class (energy must be bounded) we must have that $\lambda = -1/2$. This condition is always true for this crack problem, whether it is mode I, mode II or mode III. With that condition, the second of the above conditions is automatically satisfied ($\cos -\frac{\pi}{2} = 0$) so we must have that $A_2 + B_2 = 0$

So χ has the form

$$\chi = -r^{3/2} A_2 \left(\sin\frac{\theta}{2} - \sin\frac{3\theta}{2}\right)$$

We may now evaluate the stresses using the Airy stress function relations:

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\chi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\chi}{\partial\theta^2}, \quad \sigma_{\theta\theta} = \frac{\partial^2\chi}{\partial r^2}, \quad \sigma_{r\theta} = -\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial\chi}{\partial\theta})$$

In an analogous way to the mode I problem, we define $A_2 = K_{II}/\sqrt{2\pi}$, so that

$$\sigma_{\theta\theta} = -\frac{K_{II}}{\sqrt{2\pi r}}\frac{3}{4}(\sin\frac{\theta}{2} + \sin\frac{3\theta}{2})$$

$$\sigma_{rr} = -\frac{K_{II}}{\sqrt{2\pi r}}\frac{1}{4}(5\sin(\frac{\theta}{2}) - 3\sin(\frac{3\theta}{2})$$

$$\sigma_{r\theta} = \frac{K_{II}}{\sqrt{2\pi r}}\frac{1}{4}r^{-1/2}(\cos\frac{\theta}{2} + 3\cos\frac{3\theta}{2})$$

Unfortunately, we are not done yet. To make the formulas look nicer (and allow us to compare with the book), we should eliminate the terms involving $\cos\frac{3\theta}{2}$ and $\sin\frac{3\theta}{2}$ by replacing them with terms involving $\cos\frac{\theta}{2}$ and $\sin\frac{\theta}{2}$. Using the trigonometric double angle formulas you can show that:

$$\cos\frac{3\theta}{2} = \cos\frac{\theta}{2}\left(\cos^2\frac{\theta}{2} - 3\sin^2\frac{\theta}{2}\right) \quad \text{and} \quad \sin\frac{3\theta}{2} = \sin\frac{\theta}{2}\left(3\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right)$$

Substitute these back into above expressions for the stresses and simplify (you can simplify in different ways; it helps to take a look at the expressions in the book so you simplify in a way to get the same answer)

$$\sigma_{rr} = \frac{K_{II}}{\sqrt{2\pi r}}\sin\frac{\theta}{2}(1 - 3\sin^2\frac{\theta}{2})$$

$$\sigma_{\theta\theta} = -\frac{K_{II}}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$\sigma_{r\theta} = \frac{K_{II}}{\sqrt{2\pi r}}\cos\frac{\theta}{2}(1 - 3\sin\frac{\theta}{2})$$

Which agrees with the expressions in the book.

3. The stress intensity factor for this crack configuration is given by

$$K = \frac{1.12}{\sqrt{Q}}\sigma\sqrt{\pi a}$$

We approximate the cylinder as a thin-walled pressure vessel (90 mm inner diameter and a 110 mm outer diameter) so in this case the appropriate stress is the hoop stress, $\sigma_{\theta\theta} = Pr/t = 4.5P$. The factor Q is given in section A.7 as (include the plasticity correction)

$$Q \approx \Psi^2 - 0.212\left(\frac{\sigma}{\sigma_y}\right)^2$$

And we know that

$$\Psi \approx \frac{3\pi}{8} + \frac{\pi}{8}\left(\frac{a^2}{c^2}\right) \approx 1.353$$

When the values $a = 1.5$ mm and $c = 2.25$ mm are substituted in. So the expression for K becomes:

$$K = \frac{1.12\sigma\sqrt{\pi a}}{\sqrt{1.830 - 0.212(\sigma/\sigma_y)^2}}$$

To find the value when the crack can grow by fast fracture, set $K = K_{Ic}$. Plugging in the values $\sigma_y = 550$ MPa, $K_{Ic} = 30$ MPa$\sqrt{\text{m}}$ and $a = 1.5$ mm, and performing some simplifications find that

$$0.710 = \frac{(\sigma/\sigma_y)}{\sqrt{1.830 - 0.212(\sigma/\sigma_y)^2}}$$

Use the above equation to solve for σ/σ_y. I found that $\sigma/\sigma_y = 0.912$ or $\sigma = 502$ MPa and thus $P = \sigma/4.5 = 111$ MPa. Thus the pressure reached a value more than double the maximum intended pressure.

4. (a) Note that this problem is very similar to the worked example on pp. 312-313. For the test to be a valid measurement all the dimensions must be at least 25 times larger that the size of the plastic zone, or

$$a, (W - a), B \geq 25r_p = 25 \times \frac{1}{3\pi}\left(\frac{K_{Ic}}{\sigma_y}\right)^2$$

These conditions come from the fact that in order for our K field to be valid, which is an elasticity solution, the extent of plasticity must be small compared to the in-plane dimensions of the dimensions of the sample, and in order for the problem to be one of plane strain the dimension B must be significantly larger than the plastic zone. Substitute in the given values of we find that $25 \times r_p = 5.97$ cm. Since we are given that $a/w = 0.45$, setting $a = 6.0$ cm ensures that $W - a$ satisfies the above condition. So we have that $a = B = 6.0$ cm, $W = 13.33$ cm.

(b) Now calculate the volume and mass of each sample type. For the compact specimen the volume of the sample is approximately given by $HBW = 1.2WBW$ and thus the mass (assume the density of steel is 7.8 g/cm^3) is 10.0 kg or $\approx$ 25 lbs. For the bend specimen the volume is $B \times W \times 4W$, and thus the mass is 33.1 kg, or $\approx$ 72 lbs. In both cases we neglect the weight lost due to the starter notch.

(c) Calculate the loads from the expressions given in the appendix for these geometry, i.e. for the compact specimen we have

$$K_I = \frac{P}{BW^{1/2}} f(\frac{a}{W})$$

Where $f = 8.35$ for $a/W = 0.45$. The load P at failure corresponds to $K_I = K_{Ic}$ so we have that

$$P = 0.12 K_{Ic} B\sqrt{W} = 394 \times 10^3 \text{ N}$$

and for the bend specimen

$$K_I = \frac{PS}{BW^{3/2}} f(\frac{a}{W})$$

Where $f = 2.28$ for $a/W = 0.45$. Also, set the span S equal to $3W$. The load P at failure corresponds to $K_I = K_{Ic}$ so we have that

$$P = 0.15 K_{Ic} B \sqrt{W} = 360 \times 10^3 \text{ N}$$

Both are larger than our available test machine capacity of 200 kN.

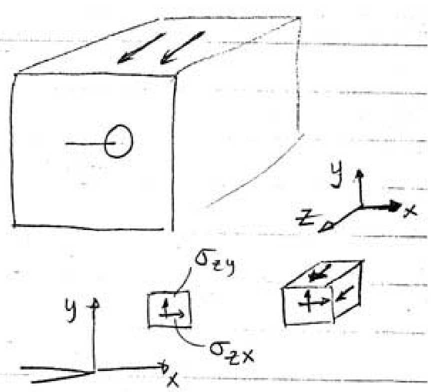
(d) We find that the specimen size requirements necessitate a very large sample, and the load requirements are larger than the available test machine. This is generally the case for ductile materials which have large plastic zones. Even by cleverly changing the specimen dimensions (i.e a/W) the load requirements are too high. One possibility is to test at low temperatures. As the temperature is decreased, both K_{Ic} and σ_y decrease but generally K_{Ic} decreases faster than σ_y so the plastic zone size (and thus the required specimen dimensions) goes down.

What is generally done for these type of materials is to use fracture testing techniques based on J testing. The specimen size requirements are much less severe, so that reasonably sized specimens can be used. The applicability of J testing has been verified by comparisons between enormous specimens (that satisfy the strict size requirements) to get a K_{Ic} value and compared with the values on small scale J tests, and if the tests are done properly the results have shown that J testing is a excellent method for obtaining the fracture toughness of ductile, tough materials.

5. Mode III

$$u_z = u_z(x, y) \quad \text{— (31)}$$

$$\begin{pmatrix} \sigma_{xz} \\ \sigma_{yz} \end{pmatrix} = 2G \begin{pmatrix} \varepsilon_{xz} \\ \varepsilon_{yz} \end{pmatrix} = 2G \begin{pmatrix} \frac{\partial u_z}{\partial x} \\ \frac{\partial u_z}{\partial y} \end{pmatrix} \quad \text{— (32)}$$



Eqm $\quad \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} = 0 \quad$ — (33)

Subst (32) into (33)

$$\frac{\partial}{\partial x}\left(G\frac{\partial u_z}{\partial x}\right) + \frac{\partial}{\partial y}\left(G\frac{\partial u_z}{\partial y}\right) = 0$$ Harmonic Eqn

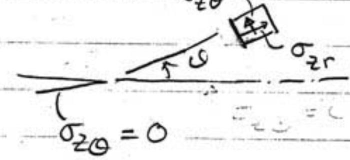
$$\Rightarrow \quad G\nabla^2 u_z = 0 \qquad (34)$$ Laplace's Eqn

Because of simple form of this eqn, we solve in terms of displ (rather than stress fcn)

Assume soln of form:

$$u_z = r^{\lambda+1}\left(A\sin(\lambda+1)\theta + B\cos(\lambda+1)\theta\right)$$ this satisfies harmonic eqn

B.C. on $\theta = \pm\pi$, $\sigma_{z\theta} = 0$



$$\sigma_{z\theta} = \frac{G}{r}\frac{\partial u_z}{\partial \theta}$$

$$\sigma_{z\theta} = G r^{\lambda}\left[A(\lambda+1)\cos(\lambda+1)\theta - B(\lambda+1)\sin(\lambda+1)\theta\right]$$

$\theta = \pi \qquad -A\cos\lambda\pi + B\sin\lambda\pi = 0 \qquad$ (a) — (35)

$\theta = -\pi \qquad -A\cos\lambda\pi - B\sin\lambda\pi = 0 \qquad$ (b)

(a)+(b) $\Rightarrow \quad A\cos\lambda\pi = 0$

(a)−(b) $\qquad B\sin\lambda\pi = 0$

1) $\cos\lambda\pi = 0 \quad \Rightarrow \lambda = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$

$B = 0$

2) $\sin\lambda\pi = 0 \quad \Rightarrow \lambda = -2, -1, 0, 1$

$A = 0$

Dominant term is $\lambda = -\frac{1}{2}$

$$u_z = r^{\frac{1}{2}} A\sin\frac{\theta}{2} + \text{hot} \qquad (36)$$

$$u_z = \frac{K_{III}}{G}\sqrt{\frac{2r}{\pi}}\sin\frac{\theta}{2} + \text{hot} \qquad (37)$$

$$\begin{Bmatrix}\sigma_{\theta z}\\ \sigma_{rz}\end{Bmatrix} = \frac{K_{III}}{\sqrt{2\pi r}}\begin{Bmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{Bmatrix} \qquad (38)$$