

# 3.21 Lectures on Fluid Flow and Kinetics

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## Abstract

In the processing-structure-properties triad which traditionally defines Materials Science and Engineering, the processing-structure link is governed largely by kinetics of reactions and phase transformations, and also heat and mass transfer. Very often the limiting step in a process is getting sufficient quantities of a reagent or heat to the right place, and very often that reagent or heat is carried by a fluid such that the rate of mass or heat transport is determined by fluid flow. This is naturally a more complex phenomenon than mass or heat transport in a solid, as fluids involve both diffusive and convective transport, the latter of which couples the scalar concentration and temperature fields to the vector velocity field. Also, the nonlinearity in the convective momentum transport terms leads to the phenomenon of turbulence, which is inherently convective in nature but is usually modeled as an enhancement to mass, heat and momentum diffusivity.

This short four-lecture series on fluid dynamics will introduce the concepts of momentum transfer, convective transport, closed-form equations for fluid behavior (the Navier-Stokes equations), coupled flow and solute/thermal diffusion in a fluid, and hopefully turbulent flow and transport. They will conclude with calculation of heat and mass transfer coefficients between a fluid and solid with simple geometry and flow behavior, with sufficient background in the concepts to understand extension to complex flows and geometries. This will complete your understanding of kinetics of processes which are limited by transport in a fluid, and round out your overall kinetics education in 3.21.

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# 1 Momentum Diffusion, Shear Stress, Pressure, and Stokes Flow

Viscous shear is a diffusive process, in that the rate of momentum transfer by viscous shear is proportional to the momentum gradient. In this way, it is similar to solute diffusion and thermal diffusion, with which you are by now very familiar. This section will begin with a brief review of solute and thermal diffusion, and introduce the concepts of momentum diffusion, the shear stress tensor convention, and the role of pressure. It concludes with the closed system of equations for Stokes Flow, which are a set of partial differential equations, one scalar and one vector, which describe the scalar pressure and vector velocity fields in low-velocity fluid flow systems.<sup>1</sup>

## 1.1 Brief review of solute and thermal diffusion

In solute diffusion, there is a concentration field  $C(x, y, z)$ , typically measured in moles (or grams) per unit volume, and flux  $\vec{J}$  given in moles per unit area per unit time, and we can write the conservation equation:

$$\frac{\partial C}{\partial t} = -\nabla \cdot \vec{J} + G, \tag{1}$$

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<sup>1</sup>Stokes flow is valid when convective transport of momentum is negligible, as discussed in section 2.3.

where  $G$  is a source term due to chemical, nuclear, or other generation of the species. This is closed by the constitutive equation given in Fick's first law, which gives the flux as proportional to the negative concentration gradient, with proportionality constant  $D$  called the diffusivity:

$$\vec{J} = -D\nabla C. \quad (2)$$

This leads straightforwardly to a closed-form governing equation for concentration:

$$\frac{\partial C}{\partial t} = \nabla \cdot D\nabla C + G, \quad (3)$$

which for constant  $D$  simplifies to

$$\frac{\partial C}{\partial t} = D\nabla^2 C + G. \quad (4)$$

In thermal diffusion, the field variable is temperature  $T(x, y, z)$ , and flux  $\vec{q}$  is given in thermal energy per unit area per unit time. Right away, this makes the conservation equation less straightforward, due to the requirement of conversion from thermal energy to temperature units; for constant pressure this becomes:

$$\frac{\partial H}{\partial t} = \rho c_p \frac{\partial T}{\partial t} = -\nabla \cdot \vec{q} + \dot{q}, \quad (5)$$

where again there's a source term  $\dot{q}$ . This is again closed by a diffusive constitutive equation, with proportionality constant  $k$  called the thermal conductivity:

$$\vec{q} = -k\nabla T. \quad (6)$$

For convenience, with constant  $k$  we can define the thermal diffusivity  $\alpha = k/\rho c_p$ , which gives a closed-form partial differential equation governing temperature which looks a lot like equation 4:

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T + \frac{\dot{q}}{\rho c_p}. \quad (7)$$

## 1.2 The velocity vector field and momentum diffusion tensor

Fluid flow starts off different from these two because the field variable is a vector field  $\vec{u}(x, y, z)$ , the fluid velocity. But like solute and thermal transport, we can write a conservation equation and a constitutive equation for each component which is diffusive. The momentum flux, which describes transfer of each component of momentum in each direction, is a second-rank tensor  $\tau$ , for example  $\tau_{yx}$  is the rate of  $x$ -momentum transfer in the  $y$ -direction. As with thermal diffusion, this complicates the conservation equation just a bit, but momentum density is easy enough to write as  $\rho\vec{u}$ , so for example the  $x$ -component of the conservation equation goes like:

$$\frac{\partial(\rho u_x)}{\partial t} \text{ shear} = -\frac{\partial\tau_{xx}}{\partial x} - \frac{\partial\tau_{yx}}{\partial y} - \frac{\partial\tau_{zx}}{\partial z}. \quad (8)$$

Equation 8 is not at all complete, it merely describes the shear stress contribution to the rate of  $x$ -momentum accumulation.

## 1.3 Relationship between momentum diffusion and mechanical stress

Speaking of shear stress, this "momentum flux" can be thought of as a stress because of Newton's law:

$$\vec{F} = m\vec{a} = \frac{d(m\vec{v})}{dt}, \quad (9)$$

Momentum accumulation rate is thus equal to force, and momentum flux rate per unit area is a force per unit area, or a shear stress. And like the mechanical stress, this is a symmetric tensor:  $\tau_{yx} = \tau_{xy}$ . But there are two important differences between the momentum flux/shear stress tensor  $\tau$  and the mechanical stress tensor  $\sigma$ :

1. The sign conventions for mechanical stress and momentum diffusion are opposite:

$$\sigma_{xy} = -\tau_{xy}. \quad (10)$$

This is because in primitive times, the mechanical and chemical engineers dwelt in different caves and didn't communicate except by smoke signals, and it's hard to represent equations in smoke signals. So for their separate reasons they came up with different ways of saying the same thing, and as materials engineers who deal with both solid mechanics and fluid flow, we are forced to negotiate the two different conventions. The mechanical engineers wanted to express tension as a positive stress and compression as a negative stress, and for consistency in their constitutive equation, express greater  $x$ -force on top of the box than on the bottom as positive shear stress  $\sigma_{yx} > 0$ . The chemical engineers say that larger  $x$ -force on the bottom than the top leads to diffusion of  $x$ -momentum in the positive  $y$ -direction, and positive  $\tau_{yx}$ . Note that this means that positive shear stress in the  $x$ -direction  $\tau_{xx} > 0$  implies compression. This is illustrated in figure 1.

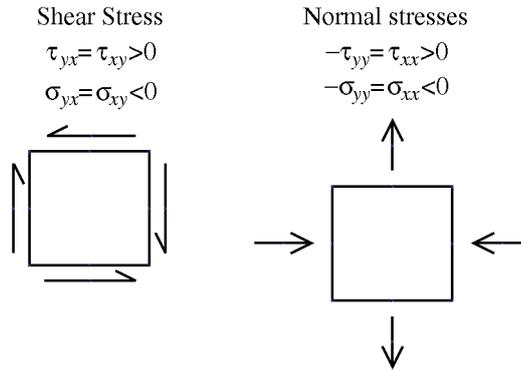


Figure 1: Sign conventions for stress states.

2. The shear stress  $\tau$  has no deviatoric component (that is, no net pressure), so its diagonals sum to zero:

$$\tau_{xx} + \tau_{yy} + \tau_{zz} = 0. \quad (11)$$

For this reason, the shear stress  $\tau$  has one less degree of freedom than the mechanical stress  $\sigma$  (five vs. six in three dimensions, two vs. three in two), and we must complement  $\tau$  with the pressure  $p$ , which is equal to the average of the diagonal mechanical stresses, with opposite sign of course just to keep things interesting:

$$p = -\frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3}. \quad (12)$$

The pressure is an additional scalar field in the fluid system.

These two differences are summarized by the relationship between fluid dynamics shear stress and mechanical stress:

$$\sigma_{ij} = -\tau_{ij} - pI, \quad (13)$$

where  $I$  is the identity matrix.

## 1.4 Stokes flow

The pressure contributes to momentum accumulation in that it provides an additional driving force down the pressure gradient, so we can add that to equation 8 to give a nearly complete equation for conservation of  $x$ -momentum:

$$\frac{\partial(\rho u_x)}{\partial t} = -\frac{\partial p}{\partial x} - \frac{\partial \tau_{xx}}{\partial x} - \frac{\partial \tau_{yx}}{\partial y} - \frac{\partial \tau_{zx}}{\partial z}. \quad (14)$$

All three components can be written by expressing this in vector form:

$$\frac{\partial(\rho\vec{u})}{\partial t} = -\nabla p - \nabla \cdot \tau + \vec{F}, \quad (15)$$

where the final term  $\vec{F}$  represents body forces such as gravity, Lorentz force, etc. which also lead to momentum accumulation. This is the equation of motion for *Stokes flow*, which is valid for low velocities or high viscosities as will be discussed in the next lecture.

Note that equation 15 is not closed, since it represents a single vector equation for the vector field  $\vec{u}$ , the tensor field  $\tau$ , and the scalar fields  $p$  and  $\rho$ . In three dimensions, this has three equations and seven unknown fields. We can eliminate five of those fields by expressing shear stress  $\tau$  in terms of the velocity in the constitutive equation, another by an expression for  $\rho$ , and provide a final closing equation called the equation of continuity using the principle of mass conservation, these three will be done in that order below.

#### 1.4.1 Constitutive equation for $\tau$

In terms of  $\tau$ , the analogy can be made again to solute and thermal diffusion constitutive relations in equations 2 and 6. A Newtonian fluid exhibits a linear relationship between shear stress and velocity gradient, with the proportionality constant  $\mu$  called the viscosity:<sup>2</sup>

$$\tau_{yx} = -\mu\dot{\gamma}_{yx} = -\mu \left( \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right), \quad (16)$$

where  $\dot{\gamma}$  is the shear strain rate tensor. Note that whereas mechanics involves the relationship between stress and strain, fluid dynamics deals with stress and strain *rate*, since a fluid by definition has zero yield stress; this is why the elastic equations lead to wave-like behavior and the fluid equations to diffusive behavior. The two partial derivatives in parentheses are there to keep the shear stress and strain rate symmetric. But the point remains: momentum diffusion flux is proportional to the negative velocity gradient, just as solute diffusion flux is proportional to the negative concentration gradient, and thermal flux to negative temperature gradient. The diagonal stresses have one more term which we won't go into, but which appears in the overall vector form of the constitutive equation:

$$\tau = -\mu \left( \nabla\vec{u} + \nabla\vec{u}^T - \frac{2}{3}\nabla \cdot \vec{u}I \right), \quad (17)$$

where  $T$  denotes the transpose of the second velocity gradient outer product, again to keep things symmetric.

The shear strain is dimensionless, but the shear strain rate dimension is inverse time, so the viscosity has dimensions of stress $\times$ time. In SI units, this is  $\frac{\text{N}\cdot\text{s}}{\text{m}^2} = \frac{\text{kg}}{\text{m}\cdot\text{s}}$ , in cgs units we define the poise as  $1 \frac{\text{erg}\cdot\text{s}}{\text{cm}^2}$ . Water has a viscosity of 0.01 poise, or one centipoise, for this reason many fluid viscosities are tabulated in units of centipoise; in SI units this is  $0.001 \frac{\text{N}\cdot\text{s}}{\text{m}^2}$ .

#### 1.4.2 Expressions for density $\rho$

The density  $\rho$  is generally a function only of the pressure  $p$ , for example in gases the ideal gas equation gives a good approximation of density:

$$\rho = \frac{n}{V} = \frac{p}{RT}. \quad (18)$$

In liquids and low-velocity gas flows (below about Mach 0.3 and with pressure variations less than about 10% of the average absolute pressure), the functionality is sufficiently weak that the density can be taken as a constant. This is a sufficient (but not necessary) condition for *incompressible flow*, which will be defined more rigorously in section 2.2.1.

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<sup>2</sup>Many fluids do not exhibit this linear relationship between shear stress and velocity gradient/shear strain rate, and are thus non-Newtonian fluids with more complex constitutive equations. Examples include polymers and dense particle suspensions in liquids, whose apparent viscosity decreases with increasing shear rate, and dilute particle suspensions, whose apparent viscosity can increase with increasing shear rate, particularly if the particles have high aspect ratio. But detailed treatment of nonlinear fluids is beyond the scope of this brief introduction.

### 1.4.3 Mass conservation and the continuity equation

Unfortunately, unlike density and shear stress, there is no simple way to express pressure as a function of the other variables, and “constant pressure” is not an option. But there is an additional conservation law which provides us with the final equation to close the Stokes flow system, which is conservation of mass. The mass flux is purely convective, and equal to the product of velocity and density  $\rho\vec{u}$  (that this is also the momentum density is somewhat coincidental). A density gradient does not by itself lead to transport, as phase boundaries and other factors can lead to nonuniform density in an equilibrium system; put slightly differently, there is no “mass diffusion”. The equation for mass conservation, also known as the *continuity equation*, sets mass accumulation rate equal to the divergence of convective mass flux:

$$\frac{\partial\rho}{\partial t} = -\nabla \cdot (\rho\vec{u}). \quad (19)$$

This is traditionally written with all of the terms expanded on the left side:

$$\frac{\partial\rho}{\partial t} + \vec{u} \cdot \nabla\rho + \rho\nabla \cdot \vec{u} = 0. \quad (20)$$

If flow is incompressible, the first two terms are zero,<sup>3</sup> and we’re left with

$$\nabla \cdot \vec{u} = 0. \quad (21)$$

### 1.4.4 The simplified closed Stokes flow equations

For a Newtonian fluid with constant  $\mu$  and  $\rho$ , we can rewrite equation 15, the equation of motion, as follows:

$$\rho\frac{\partial\vec{u}}{\partial t} = -\nabla p + \mu\left[\nabla^2\vec{u} + \nabla \cdot (\nabla\vec{u}^T) - \frac{2}{3}\nabla \cdot (\nabla \cdot \vec{u}I)\right] + \vec{F}. \quad (22)$$

When flow is incompressible,  $\nabla \cdot \vec{u} = 0$ , so the second and third terms in the shear strain divergence are zero; dividing what’s left by  $\rho$  leaves us with:

$$\frac{\partial\vec{u}}{\partial t} = -\frac{\nabla p}{\rho} + \nu\nabla^2\vec{u} + \frac{\vec{F}}{\rho}. \quad (23)$$

The parameter  $\nu = \mu/\rho$  is the *kinematic viscosity*, also known as the momentum diffusivity. It is analogous to the thermal diffusivity  $\alpha = k/\rho c_p$ , in that it also leaves us with the equation in a form looking very much like the diffusion equation. Again using water as an example, its kinematic viscosity is  $\frac{0.001\text{kg}/(\text{m}\cdot\text{s})}{1000\text{kg}/\text{m}^3} = 10^{-6}\text{m}^2/\text{s}$ .<sup>4</sup>

Equations 21 and 23 constitute the Stokes flow equations for an incompressible Newtonian constant-viscosity fluid.

## 1.5 Summary of solute, thermal and momentum diffusion

Table 1 summarizes the equations of solute, thermal and momentum diffusion (parenthesized numbers are equation references in the above text).

## 1.6 Examples

### 1.6.1 Couette flow

Couette flow is simplified flow between two parallel plates, with one plate moving, and the other stationary. For example, consider two large horizontal plates with a 1 mm layer of water between them, the bottom plate stationary and the top one moving at 1 cm/s. We’ll discuss two situations: the steady-state which the system reaches after a “long time” (which will be defined), and the transient behavior starting from a motionless fluid as it evolves to the steady-state solution.

<sup>3</sup>It is easy to see this for constant density because all of the  $\rho$  derivatives will be zero, but  $\partial\rho/\partial t + \vec{u} \cdot \nabla\rho = 0$  is a somewhat weaker condition than constant density, as will be discussed in section 2.2.1.

<sup>4</sup>Coincidentally, many molten metals have kinematic viscosity very close to this value, so one can build a physical model of molten metal flow using water and measure velocities very close to what one would expect in the molten metal. Even the kinematic viscosity of air at 300K is  $\frac{10^{-5}\text{kg}/(\text{m}\cdot\text{s})}{1.9\text{kg}/\text{m}^3} = 5 \times 10^{-6}\text{m}^2/\text{s}$ , which is still within an order of magnitude of water.

Equation	Solute Diffusion	Thermal Diffusion	Momentum Diffusion
Conservation	$\frac{\partial C}{\partial t} = -\nabla \cdot \vec{J} + G$ (1)	$\rho c_p \frac{\partial T}{\partial t} = -\nabla \cdot \vec{q} + \dot{q}$ (5)	$\frac{\partial(\rho \vec{u})}{\partial t} = -\nabla p - \nabla \cdot \tau + \vec{F}$ (15)
Constitutive	$\vec{J} = -D \nabla C$ (2)	$\vec{q} = -k \nabla T$ (6)	$\tau = -\mu (\nabla \vec{u} + \nabla \vec{u}^T - \frac{2}{3} \nabla \cdot \vec{u} \vec{I})$ (17)
Diffusivity	$D = D$	$\alpha \equiv k / \rho c_p$	$\nu \equiv \mu / \rho$
Diffusion equation <sup>5</sup>	$\frac{\partial C}{\partial t} = D \nabla^2 C + G$ (4)	$\frac{\partial T}{\partial t} = \alpha \nabla^2 T + \frac{\dot{q}}{\rho c_p}$ (7)	$\frac{\partial \vec{u}}{\partial t} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \vec{u} + \frac{\vec{F}}{\rho}$ (23)

Table 1: Comparison of solute, thermal and momentum diffusion.

**Steady state** At steady state, the time derivative of velocity is zero. As long as there is no body force or pressure gradient in the direction of travel, at steady-state equation 23 reduces to

$$\nabla^2 \vec{u} = 0, \quad (24)$$

and if flow is only in the direction of motion of the upper plate (here the  $x$ -direction) and it varies only in the direction normal to the plane (here the  $y$ -direction), this reduces to the trivially simple ordinary differential equation

$$\frac{d^2 u_x}{dy^2} = 0, \quad (25)$$

with the obvious solution

$$u_x = Ay + B. \quad (26)$$

This is the steady-state momentum diffusion solution in 1-D, and is identical in form to the steady-state solute or thermal diffusion solution.

In diffusion, the objective is often the flux; here the flux is the shear stress, the force per unit area required to maintain this flow, given here by:

$$\tau_{yx} = -\mu \frac{\partial u_x}{\partial y} = -10^{-3} \frac{\text{N} \cdot \text{s}}{\text{m}^2} \frac{0.01 \text{m/s}}{0.001 \text{m}} = -0.01 \frac{\text{N}}{\text{m}^2}. \quad (27)$$

Note the negative sign of the shear stress, this is because the  $x$ -momentum is diffusing in the negative  $y$ -direction. One must thus exert a force in the positive  $x$  direction on the top plate, and the negative  $x$  direction on the bottom plate, to maintain this flow field.

**Transient behavior** Now consider the transient behavior during the start of flow. At time  $t = 0$ , both plates and the fluid are stagnant, and the top plate starts moving at a speed of 1 cm/s in the  $x$ -direction. Again assuming no pressure gradients or body forces in the  $x$ -direction, equation 23 becomes:

$$\frac{\partial u_x}{\partial t} = \nu \frac{d^2 u_x}{dy^2}. \quad (28)$$

This is basically the diffusion equation, and with the uniform-velocity initial condition and constant-velocity boundary condition at  $y = 1\text{mm}$ , the solution at short times (over which the fluid can be considered semi-infinite) is:

$$u_x = 1 \frac{\text{cm}}{\text{s}} \times \text{erfc} \left( \frac{1\text{mm} - y}{2\sqrt{\nu t}} \right). \quad (29)$$

This solution is valid for as long as it gives approximately zero velocity at  $y = 0$ . Since  $\text{erfc}(2) = 0.005$ , we can take that as our semi-infinite criterion, and set

$$\frac{1\text{mm} - y}{2\sqrt{\nu t}} = 2 \text{ at } y = 0, \quad (30)$$

which when solved for time  $t$  gives

$$t = \frac{(1\text{mm})^2}{16\nu}. \quad (31)$$

The erfc solution is valid until this time.

Note that the form of this timescale is a length squared divided by the momentum diffusivity, which again is just like the solute diffusion timescale. In fact, as with solute or thermal diffusion, we can use the “long times” expression of

$$t \sim \frac{L^2}{\nu} \quad (32)$$

to approximate the time at which the steady-state solution above can be considered valid, in this case this is  $(0.001\text{m})^2/(10^{-6}\text{m}^2/\text{s}) = 1$  second.

Also note that at  $t = 0$ , the singularity in the initial and boundary conditions leads to a singularity in the shear stress. The derivative of the erfc at  $y = 1$  mm is infinite at  $t = 0$ , thus requiring an infinite shear stress to achieve a step jump in velocity from zero to 1 cm/s. The shear stress goes as  $1/\sqrt{t}$ , so the time integral and thus impulse per unit area is finite, and as long as the motor can provide a very large force relative to the steady state, the boundary condition can approximately be satisfied, and the erfc solution is roughly valid.<sup>6</sup>

### 1.6.2 Gravity-driven flow down an inclined plane

(This section, when written, will pretty much follow the treatment in chapter 2 of D.R. Poirier and G.H. Geiger, *Transport Phenomena in Materials Processing*.)

### 1.6.3 Flow past a sphere

A solid sphere of radius  $R$  moves through a fluid at constant velocity  $V_\infty$  relative to the fluid. In the frame of reference of the sphere, it is stationary, and the fluid far from it is moving at that relative velocity, though the fluid close to it is moving more slowly, and the fluid at its surface is at zero velocity.

In spherical coordinates, where  $r$  is the distance from the center of the sphere,  $\phi$  is the angle from the flow field axis of symmetry (so  $\phi = \pi$  at the stagnation point), and  $\theta$  is the angle measured around the sphere, the steady-state flow velocity and pressure in the frame of reference of the sphere are given by:

$$u_r = V_\infty \left( 1 - \frac{3R}{2r} + \frac{1}{2} \left( \frac{R}{r} \right)^3 \right) \cos \phi \quad (33)$$

$$u_\phi = -V_\infty \left( 1 - \frac{3R}{4r} - \frac{1}{4} \left( \frac{R}{r} \right)^3 \right) \sin \phi \quad (34)$$

$$p = p_0 - \rho gh - \frac{3}{2} \mu \frac{V_\infty}{R} \left( \frac{R}{r} \right)^2 \cos \phi, \quad (35)$$

where the gravitational force acts in the negative  $h$  direction, and  $p_0$  is the far-field pressure at  $h = 0$ .

This information can be used to calculate the drag force exerted on the sphere by the fluid, and vice versa. The normal component of the total stress is the *traction*  $\vec{t}$ , this can be written

$$\vec{t} = \tau \cdot \hat{n} + p\hat{n}, \quad (36)$$

this is the force per unit area exerted by the solid surface on the fluid. The integral of this traction over the surface gives the total force exerted by the solid on the fluid, so with opposite sign it is the *drag force* exerted on the solid:

$$\vec{F}_d = - \int_{\text{sphere surface}} (\tau \cdot \hat{n} + p\hat{n}) dA. \quad (37)$$

In this case, the integral over the surface of the sphere with  $\tau_{r\theta} = 0$  and  $\partial\tau/\partial\theta = 0$  is given by:

$$\vec{F}_d = - \int_{\phi=0}^{\pi} (\tau_{rr}\hat{r} + \tau_{r\phi}\hat{\phi} + p\hat{r}) dA = 6\pi\mu RV_\infty \hat{z}, \quad (38)$$

where  $\hat{z}$  is the  $\phi = 0$  direction, the direction of flow.

<sup>6</sup>Thanks to Adam Nolte for thin finite impulse explanation.

## 1.7 Exercises

1. Show that the velocity and pressure fields for Stokes flow past a sphere given in section 1.6.3 (equations 33 through 35) satisfy the Stokes flow equations in spherical coordinates.
2. Show that the shear stress and pressure at the surface of the sphere at  $r = R$  given by the velocity field in section 1.6.3 (equations 33 through 35) integrate to give the total drag force given by equation 38.
3. Set the total drag force for flow past a sphere given in equation 37 equal to the buoyancy force (fluid-solid sphere density difference times the sphere volume), and solve for velocity to give an expression for the terminal velocity of a rising/sinking spherical particle in Stokes flow.

## 2 The Substantial Derivative and the Navier-Stokes Equations

Convective transport involves the “carrying” of solutes, heat, mass or momentum by a body moving relative to the frame of reference. In fluid systems, this typically enhances transport rates quite a bit above diffusion alone. One does not typically use the word convection to describe transport in a solid, but in many situations it is convenient to define the frame of reference in motion relative to the solid, and then the same analysis applies.

This section will begin by introducing convective transport and the substantial derivative in a solid, then discuss their application to a fluid. This will lead to the convective terms in the mass and momentum conservation equations, and to the full Navier-Stokes equations for fluid behavior. The Reynolds number describes the relative importance of the convective and diffusive terms of the equation of motion, if it is very small then we can neglect the convective terms and use the Stokes flow equations derived earlier in part 1.4. At high Reynolds number, the nonlinear convective terms give rise to a flow instability which leads to turbulence. These will be discussed over the course of this section.

### 2.1 The substantial derivative

In a substance not moving with respect to our frame of reference, the diffusion equation looks like equations 1 through 4. The partial derivative on the left with respect to time describes the change in concentration with respect to time at a fixed point in space. Working in two dimensions for now, (as extension to three is obvious and only makes things messy), the partial derivative is defined as:

$$\frac{\partial C}{\partial t} \equiv \lim_{\Delta t \rightarrow 0} \frac{C(x, y, t + \Delta t) - C(x, y, t)}{\Delta t}. \quad (39)$$

However, if the point we’re interested is moving relative to our reference frame, then instead of the partial derivative, we really want the difference between the state at the new time and location and the original time and location. Thus we define the substantial derivative as:

$$\frac{DC}{Dt} \equiv \lim_{\Delta t \rightarrow 0} \frac{C(x + \Delta x, y + \Delta y, t + \Delta t) - C(x, y, t)}{\Delta t}. \quad (40)$$

Neglecting the terms beyond first order gives:

$$\frac{C(x + \Delta x, y + \Delta y, t + \Delta t) - C(x, y, t)}{\Delta t} \simeq \frac{\Delta C}{\Delta x} \frac{\Delta x}{\Delta t} + \frac{\Delta C}{\Delta y} \frac{\Delta y}{\Delta t} + \frac{\Delta C}{\Delta t}. \quad (41)$$

The velocity of a moving particle is defined as:

$$u_x = \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t}, \quad u_y = \lim_{\Delta t \rightarrow 0} \frac{\Delta y}{\Delta t}. \quad (42)$$

Therefore, when we take the limit as  $\Delta t$  goes to zero in equation 41, this results in:

$$\frac{DC}{Dt} = \frac{\partial C}{\partial t} + u_x \frac{\partial C}{\partial x} + u_y \frac{\partial C}{\partial y} = \frac{\partial C}{\partial t} + \vec{u} \cdot \nabla C. \quad (43)$$

This substantial derivative will replace the partial time derivative in our transport equations.<sup>7</sup>

Another way to look at this is in terms of the convective flux of a solute. The amount of a solute transported per unit area and per unit time is simply given by the product of velocity and concentration  $\vec{u}C$ . The rate of accumulation due to convective flux is therefore the negative divergence of this flux,  $-\nabla \cdot (\vec{u}C)$ . When this is added to equation 4, we get:

$$\frac{\partial C}{\partial t} = -\nabla \cdot (\vec{u}C) + D\nabla^2 C + G. \quad (44)$$

Moving the convective terms to the left side and expanding gives:

$$\frac{\partial C}{\partial t} + \vec{u} \cdot \nabla C + C\nabla \cdot \vec{u} = D\nabla^2 C + G. \quad (45)$$

Note that the first two terms on the left are the substantial derivative. The third term goes to zero in an incompressible fluid, so if that assumption is satisfied, equation 45 reduces to:

$$\frac{DC}{Dt} = D\nabla^2 C + G. \quad (46)$$

### 2.1.1 Example: heat conduction, convection, and melting

Consider a solid rod initially at temperature  $T_0$  which is heated at one end and begins to melt. We will consider the heating to be uniform so the melt interface is planar and perpendicular to the cylinder's axis (perhaps it is rotating, or is vertical like a candle or consumable arc remelting electrode). This constitutes a moving boundary problem, in that the melt interface represents a boundary condition whose location is changing with time.

The analysis of this situation may be greatly simplified if we shift the frame of reference to that of the melt interface. In this frame of reference, the boundary is fixed, but the solid is moving into the melt interface at a uniform velocity (neglecting changes in density due to thermal expansion). The convective flux of heat is somewhat difficult to give in an absolute sense, but if we're interested in differences in convective flux and not its absolute magnitude, we can use (at constant pressure)  $\vec{u}\rho c_p T$ .

Now consider a slice of the cylinder of thickness  $\Delta x$ , whose left face is at  $x$  and right face at  $x + \Delta x$ , where the  $x$ -axis is parallel to the cylinder's axis. Assume that heat is transferred in the axial direction alone, and temperature is a function only of  $x$ . Balancing the total heat in and out with accumulation in this slice, we obtain:

$$A\Delta x \cdot \rho c_p \frac{\partial T}{\partial t} = [A(u_x \rho c_p T + q_x)]_x - [A(u_x \rho c_p T + q_x)]_{x+\Delta x}, \quad (47)$$

rearranging gives:

$$\rho c_p \frac{\partial T}{\partial t} = \rho c_p u_x \frac{T_x - T_{x+\Delta x}}{\Delta x} + \frac{q_x - q_{x+\Delta x}}{\Delta x}, \quad (48)$$

and when we take the limit as  $\Delta x \rightarrow 0$  and rearrange further,

$$\rho c_p \frac{\partial T}{\partial t} + \rho c_p u_x \frac{\partial T}{\partial x} = \rho c_p \frac{DT}{Dt} = -\frac{\partial q_x}{\partial x}. \quad (49)$$

There it is again on the left side, the substantial derivative. This can be simplified further with constant conductivity, essentially giving us equation 7 with the partial time derivative replaced by a substantial derivative:

$$\frac{DT}{Dt} = \alpha \frac{\partial^2 T}{\partial x^2}, \quad (50)$$

or in vector form with the source term,

$$\frac{DT}{Dt} = \alpha \nabla^2 T + \frac{\dot{q}}{\rho c_p}. \quad (51)$$

---

<sup>7</sup>Thanks to David Dussault (Mech.E. S.M. 2002) for this explanation of the Substantial derivative.

As the melting proceeds, the heat is conducted down the length of the rod, raising the temperature away from the melt interface. At the same time, the relative motion of the rod toward the interface pushes the heat back toward the melt interface, lowering the temperature. Over time these reach a steady state profile, with temperature only a function of  $x$ , where the derivative  $\partial T/\partial t = 0$ , so equation 50 can be written:

$$u_x \frac{dT}{dx} = \alpha \frac{d^2T}{dx^2}. \quad (52)$$

Solution of this is trivial:

$$T = A + B \exp\left(\frac{u_x x}{\alpha}\right). \quad (53)$$

It is interesting to note that a steady state solution exists for this system, and also that the competing convective and conductive heat fluxes lead to a fixed penetration depth on the order of  $\alpha/u_x$ . With the melt interface ( $x = 0$ ) at the melting point ( $T = T_m$ ), and a rod much longer than  $\alpha/u_x$ , we fit the constants so that  $A = T_0$  and  $B = T_m - T_0$ , and complete the solution of our first mixed convection/diffusion problem.

## 2.2 Convection in the fluid flow equations

Using the concept of the substantial derivative allows us to understand better the equation of continuity (equation 20), and gives us new terms in the equation of motion (equations 15 and 23).

### 2.2.1 Revisiting the continuity equation

Based on our definition of the substantial derivative, the continuity equation can be rewritten:

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{u} = 0. \quad (54)$$

One can see now that the incompressibility condition corresponds to zero substantial derivative of the density. This is a weaker assumption than constant density, because the density need not be uniform everywhere, it merely must not change in the frame of reference of a moving fluid particle.

For example, in an oil-water blend, the density of the oil and water phases is quite different, so there is not constant density. But even when stirred, each phase will maintain its respective density, so if one were to follow a fluid particle in the oil phase, the density at that moving fluid particle would not change, likewise for the water phase. This flow is therefore incompressible, and the continuity equation can be simplified to the form in equation 21.

### 2.2.2 Momentum convection

As with everything else about momentum, its convection is a bit more complicated. Recall that the convective solute flux is  $\vec{u}C$ , convective heat flux can be given as  $\vec{u}\rho c_p T$  (as long as we're taking differences in convective heat flux for small enough  $\Delta T$  that  $c_p$  is constant), and convective mass flux is  $\rho\vec{u}$ . Similarly, the convective flux of momentum is given by the outer product of velocity and momentum,  $\rho\vec{u}\vec{u}$ , which like the momentum diffusion flux (*a.k.a.* shear stress), is a second-rank tensor. Neglecting stress, pressure and body forces, this gives the momentum accumulation rate of:

$$\frac{\partial(\rho\vec{u})}{\partial t} = -\nabla \cdot (\rho\vec{u}\vec{u}). \quad (55)$$

This is a bit difficult to understand, let alone visualize, but can be illustrated using a simple example, and will be gradually clarified by other examples throughout the rest of the lecture series. Suppose a flow field is given at time  $t = 0$  by velocities:

$$u_x = 1, \quad (56)$$

$$u_y = x. \quad (57)$$

The constant  $x$ -velocity is going to move the flow field in the positive  $x$ -direction. As a result, a moment later, the shifted  $y$ -velocities at  $t = 1$  will be given by:

$$u_y = x - 1, \quad (58)$$

and at any time by:

$$u_y = x - t. \quad (59)$$

This is because

$$\frac{\partial \rho u_y}{\partial t} = -\rho \vec{u} \cdot \nabla u_y - \rho u_y \nabla \cdot \vec{u} + u_y \vec{u} \cdot \nabla \rho, \quad (60)$$

and if it's incompressible,

$$\frac{\partial \rho u_y}{\partial t} = -\rho \left( u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} \right). \quad (61)$$

Here the right side of this is  $-\rho(1 \times 1 + 0)$ , so  $\partial u_y / \partial t = -1$ , which gives the result in equation 59.

### 2.2.3 The Navier-Stokes equations

Recall that the rate of accumulation due to convective flux is the opposite of divergence, the simplest case of which gives rise to the continuity equation, as was shown in section 1.4.3. In the case of momentum, the opposite of the divergence of the convective momentum flux gives the accumulation due to convective transport, so we can add the term  $-\nabla \cdot (\rho \vec{u} \vec{u})$  to the right side of the Stokes motion equation given in section 1.4, equation 15:

$$\frac{\partial(\rho \vec{u})}{\partial t} = -\nabla \cdot (\rho \vec{u} \vec{u}) - \nabla p - \nabla \cdot \tau + \vec{F}. \quad (62)$$

Moving the convective terms to the left and expanding:

$$\rho \frac{\partial \vec{u}}{\partial t} + \vec{u} \frac{\partial \rho}{\partial t} + \rho \vec{u} \cdot \nabla \vec{u} + \vec{u} \vec{u} \cdot \nabla \rho + \rho \vec{u} \nabla \cdot \vec{u} = -\nabla p - \nabla \cdot \tau + \vec{F}. \quad (63)$$

A slight rearrangement gives:

$$\rho \left( \frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} \right) + \vec{u} \left( \frac{\partial \rho}{\partial t} + \vec{u} \cdot \nabla \rho + \rho \nabla \cdot \vec{u} \right) = -\nabla p - \nabla \cdot \tau + \vec{F}. \quad (64)$$

Note that the first set of parentheses on the left encloses the substantial derivative of velocity with respect to time. The second set of parentheses encloses the left side of equation 20, and that entire term is thus zero. This allows us to simplify the equation considerably:

$$\rho \frac{D\vec{u}}{Dt} = -\nabla p - \nabla \cdot \tau + \vec{F}. \quad (65)$$

This is the complete, general equation of momentum conservation in a fluid. This and equation 20 (or if you prefer, equation 54) comprise the Navier-Stokes equations. In an incompressible Newtonian fluid with constant viscosity, equation 65 reduces to:

$$\frac{D\vec{u}}{Dt} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \vec{u} + \frac{\vec{F}}{\rho}. \quad (66)$$

The convective velocity terms in the substantial derivative, given by  $\rho \vec{u} \cdot \nabla \vec{u}$ , are also called the inertial terms. They are the nonlinear terms of this system of equations which make the whole thing such a pain to solve. And they lead to turbulence.

The derivation of these equations represents the pinnacle of complexity of this lecture series. The remainder of the series will give examples which help to provide insight into the relationship between these equations and the physical phenomena they describe, particularly convective momentum transport, and finally couple the fluid flow equations with solute and heat transport equations including convection to describe heat and mass transfer.

### 2.2.4 Example: pressure-driven flow in a tube

This section, when written, will go through the outline given in the handout “Solving Fluid Dynamics Problems” for pressure-driven flow in a tube.

## 2.3 The Reynolds number

In essence, the Reynolds number, abbreviated Re, is an estimate of the ratio of inertial terms to convective terms. Roughly speaking, in a flow field with a characteristic velocity  $U$ , characteristic lengthscale  $L$ , and constant kinematic viscosity  $\nu$ , this is estimated as:

$$\frac{\vec{u} \cdot \nabla \vec{u}}{\nu \nabla^2 \vec{u}} \sim \frac{U \cdot \frac{U}{L}}{\nu \frac{U}{L^2}} = \frac{UL}{\nu} \equiv \text{Re}. \quad (67)$$

When the Reynolds number is less than about 0.1, we can neglect the inertial terms completely, and use the Stokes flow equations to solve for the pressure and velocity fields with good accuracy. Thus the analysis in section 1.6.3 is valid for  $\text{Re} < 0.1$ .

### 2.3.1 Example: flow past a sphere revisited

In the Stokes flow régime where  $\text{Re} < 0.1$ , drag force exerted by a fluid on a solid sphere is given by  $\vec{F} = 6\pi\mu R V_\infty \hat{z}$  (equation 37, recall  $\hat{z}$  is the direction of flow), which is proportional to viscosity, lengthscale and relative velocity.

At high Reynolds number, the convective terms take over, particularly  $\rho u_r \partial u_r / \partial r$  at the stagnation point. This helps us to define the *dynamic pressure*, which estimates the pressure difference between the upstream and downstream sides of the sphere, as

$$P_{dynamic} \equiv \frac{1}{2} \rho U^2, \quad (68)$$

where  $U$  is the characteristic velocity, here the relative sphere-far field fluid velocity  $V_\infty$ . This is similar in form to the kinetic energy, and represents the kinetic energy density, hence the use of “inertial terms” to describe the convective momentum terms.

This will give rise to a definition of the friction factor in section 3.3.

## 2.4 Origin of turbulence

The Reynolds number also helps to predict the onset of turbulent flow. Turbulent flow arises from a flow instability in shear, where a perturbation normal to the main flow direction grows into a vortex, which takes energy from the main flow. When that perturbation has a little perturbation, the shear in that vortex, in turn, gives rise to a smaller vortex, and smaller vortices form on that vortex, and on and on, until the vortex is so small that viscosity dissipates its energy.

The perturbation and vortex grow because of the inertia of the fluid, and are damped by viscous drag. The ratio of the inertial to viscous terms therefore describe the tendency for the instability to be expressed; this ratio is again the Reynolds number. For fully-developed flow in a tube, if the Reynolds number is below about 2100 (using average velocity =  $\frac{1}{2}$  maximum velocity for  $U$ , and the diameter for  $L$ ), then flow is laminar. If the Reynolds number is above 2100, then perturbations will be more likely to grow, but this does not guarantee turbulence. If entrance conditions are controlled carefully, and pressure waves due to external vibration are kept to a minimum, much larger Reynolds numbers can be achieved with laminar flow, with values as high as 100,000 achieved experimentally.

Further treatment of turbulence is beyond the scope of this brief introduction, but for a concise and in-depth introduction to turbulence, I highly recommend the half-hour video *Turbulence* and its notes in the Barker library video section, call# QC151.T8.

## 2.5 Exercises

### 1. Torsional viscometer

A device for measuring viscosity of thermoplastic resins consists of a tall cylindrical cup of inner radius  $R_2$  which holds the resin, and a cylindrical rod of radius  $R_1$  inserted downward through most of the length of the container, which is far enough from the bottom that shear below the rod is negligible. The rod is attached to a motor which turns it at a constant rate, and the cup is attached to a torque meter. The device measures torque for several rotational speeds of the rod.

Assume for your calculations that the resin is Newtonian with uniform density and viscosity and that flow in the viscometer is laminar, and focus on the velocity between the rod and cup away from the bottom.

- Sketch the device and write (or xerox) the relevant form of the Navier-Stokes equations.
- Write assumptions which allow you to simplify that horrible mess of equations, and cancel the appropriate terms in the equations above. Write the resulting simplified equations.
- Determine the general solution of the velocity and pressure in your simplified equations. (You need not solve for the pressure dependence on  $r$  and  $z$ , just show that it does not depend on  $\theta$  so you can obtain a solution for the  $\theta$ -component of velocity, which is the important one.)
- Apply the boundary conditions associated with the rod and cup, and determine the velocity profile in the viscometer, and the torque at  $r = R_2$ , as a function of  $R_1$ ,  $R_2$  and the rotational speed of the rod. Is torque a function of  $r$ ?
- Suppose the cup is 20 cm long and 8 cm in diameter, and the rod is 2 cm in diameter. At a constant rotational speed of 120 rpm, the torque is measured at  $2.0 \times 10^{-3} \text{N} \cdot \text{m}$ . What is the viscosity of the molten polymer?
- At a rotational speed of 300 rpm, the torque is twice as high. Would you say this is a Newtonian fluid? If not, is it pseudoplastic (viscosity decreases with shear strain rate) or dilatant (viscosity increases with shear strain rate)?

### 2. Simplifying the Navier-Stokes Equations (24)

Using the attached handout “Solving Fluid Dynamics Problems,” write down the assumptions you can make about each of the following flow situations.

Then starting with the relevant form of the attached Navier-Stokes equations, cancel those terms which you believe will be zero on the basis of the assumptions you have made.

- Laminar flow of water past a long horizontal cylinder, illustrated schematically in figure 2. The far-field flow velocity  $\vec{V}_\infty$  is perpendicular to the axis of the cylinder. (12)

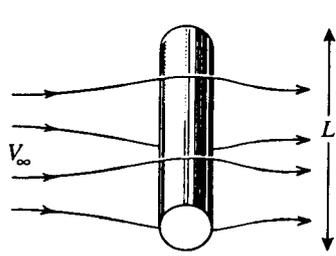


Figure 2: Schematic illustration of flow past a cylinder.

- Flow through and beyond a supersonic ( $> \text{Mach } 1$ ) Newtonian plasma torch which melts zirconia particles and sprays the molten spheres at a planar substrate perpendicular to the axis of the torch. (This is one way to make a zirconia thermal barrier coating.) You may neglect the effect of gravity on the plasma, and assume radial symmetry for flow in the torch and resulting plume.

### 3 Laminar Boundary Layers and the Friction Factor

In a great many fluid flow situations, a solid body affects the flow primarily in a localized region very close to the solid. For example, in Stokes flow around sphere (described in section 1.6.3), the sphere's effect on the overall velocity decreases as  $R/r$ . At higher Reynolds numbers, this localized region of affected flow forms a thin layer on the solid, which is known as the *boundary layer*. The traction due to this deviation in the boundary layer gives the *drag force*, so understanding the nature of the boundary layer is essential to predicting the drag force.

Likewise, in heat and mass transfer situations, the concentration or temperature is often uniform throughout most of the fluid, and deviates from that uniformity only very close to the solid surface which is adsorbing or desorbing a solute or heat. The boundary layer is thus of critical importance to heat and mass transfer as well as drag force, and we can speak of different solutal, thermal, and momentum boundary layers, which can have different structures and sizes, as we will see presently.

Section 3.1 will introduce boundary layers using a simple moving solid example.

#### 3.1 Simple example: thermal “boundary layer” in a moving solid

Suppose a thick solid sheet of a polymer is continuously extruded at temperature  $T_0$  from a die, and then cooled by fans above and below to air temperature  $T_1$ . After exiting the die, it moves with uniform velocity  $U$  in the  $x$ -direction. We would like to calculate the temperature distribution across the sheet and along its length. A high heat transfer coefficient leading to a Biot number  $hL/k$  greater than 100 ( $h$  is the heat transfer coefficient,  $L$  the sheet thickness, and  $k$  the thermal conductivity of the polymer) allows us to approximate the boundary condition as constant temperature.

Taking  $x$  as the distance from the die exit in the direction of travel, and  $y$  as the distance from the bottom face of the sheet in the direction normal to the sheet, allows us to write the steady-state heat equation with convection in the  $x$ -direction as

$$U \frac{\partial T}{\partial x} = \alpha \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right). \quad (69)$$

Where the ratio  $Ux/\alpha$  (which defines the heat transfer Peclet number) is large, thermal boundary layers will form in the top and bottom of the sheet, which are regions where the temperature is significantly lower than  $T_0$ . Letting  $\delta_T(x)$  represent the boundary layer thickness as a function of  $x$ , for  $x \gg \delta_T$ , the  $y$ -derivatives will be much larger than the  $x$ -derivatives, so we can neglect the latter in the Laplacian and write:

$$U \frac{\partial T}{\partial x} = \alpha \frac{\partial^2 T}{\partial y^2}. \quad (70)$$

Replacing  $x/U$  with  $\bar{t}$ , the time spent beyond the die exit by a polymer solid particle, recovers the 1-D transient diffusion equation, and the uniform initial and constant boundary conditions give us the erf solution:

$$\frac{T - T_1}{T_0 - T_1} = \operatorname{erf} \left( \frac{y}{2\sqrt{\alpha \bar{t}}} \right) = \operatorname{erf} \left( \frac{y}{2\sqrt{\alpha x/U}} \right). \quad (71)$$

Based on this solution, we will define the *boundary layer* as the region in which the relative temperature deviates at least 1% from the initial temperature  $T_0$ . The edge of the thermal boundary layer will be denoted  $\delta_T(x)$ , and because the error function of 1.8 is 0.99, we can set  $y = \delta_T$  where the error function argument is 1.8:

$$\frac{\delta_T}{2\sqrt{\alpha x/U}} = 1.8, \quad (72)$$

$$\delta_T = 3.6 \sqrt{\frac{\alpha x}{U}}. \quad (73)$$

Note that the ratio  $\delta_T/x$  is then a function of the Peclet number:

$$\frac{\delta_T}{x} = 3.6 \sqrt{\frac{\alpha}{Ux}} = \frac{3.6}{\sqrt{\operatorname{Pe}}}. \quad (74)$$

## 3.2 Classic fluids example: flow past a flat plate

The above heat transfer problem is a close analogue to the classic fluid dynamics problem of the velocity boundary layer in flow past a plate. In this case, an incompressible constant- $\mu$  Newtonian fluid is flowing with uniform velocity in one direction  $U_\infty$ , and a stationary plate is placed in that fluid parallel to  $U_\infty$ . Taking  $y$  as the distance normal to the plate measured from the plate surface, and  $x$  as the distance parallel to the flow measured from the leading edge of the plate, a steady-state is eventually reached with boundary conditions:

$$y \rightarrow \infty, x < 0 \Rightarrow u_y = 0, u_x = U_\infty, \quad (75)$$

$$y = 0, x \geq 0 \Rightarrow u_y = 0, u_x = 0. \quad (76)$$

This is sketched in figure 3.

Insert figure here.

Figure 3: Sketch of laminar flow boundary layers in flow past a plate.

### 3.2.1 Momentum boundary layer in flow past a plate

The Navier-Stokes equations give us a momentum diffusion boundary layer, in which velocity deviates at least 1% from  $U_\infty$ , within a region whose edge is defined by  $\delta_u(x)$ . This boundary layer is very similar in form to  $\delta_T$  above, with one important difference in the equations leading to a slight difference in the result. Like the thermal boundary layer, the  $x$ -velocity is lower in the velocity boundary layer, and reaches zero (the plate velocity) at  $y = 0$ . And again, we can examine the situation where  $x \gg \delta_u$  to neglect momentum diffusion in the  $x$ -direction.

However, unlike the thermal boundary layer, the continuity relation plays an important role here, which for an incompressible fluid in 2-D is:

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0. \quad (77)$$

Since  $u_x$  is decreasing with  $x$ ,  $u_y$  must increase with  $y$ , and we have non-negligible velocity in the  $y$ -direction. This results in convective momentum transfer away from the plate, which propels low- $x$ -velocity fluid into the stream, adding to the diffusive viscous transfer of momentum. The result is the Blasius solution, shown in figure 4, which gives dimensionless  $x$ -velocity  $u_x/U_\infty$  as a function of dimensionless distance from the plate which is labelled  $\beta$  and given by

$$\beta \equiv y \sqrt{\frac{U_\infty}{\nu x}}. \quad (78)$$

Insert figure here.

Figure 4: Blasius solution to the asymptotic Navier-Stokes equations in flow past a plate.

The Blasius solution reaches a relative velocity  $u_x/U_\infty = 0.99$  where  $\beta = 5.0$ , so this defines the edge of the boundary layer at

$$\delta_u = 5.0 \sqrt{\frac{\nu x}{U_\infty}}. \quad (79)$$

As stated above, this is very similar in form to the thermal boundary layer, given in equation 73. And here again, we can express the dimensionless boundary layer thickness  $\delta_u/x$  in terms of a dimensionless parameter, in this case the Reynolds number:

$$\frac{\delta_u}{x} = 5.0 \sqrt{\frac{\nu}{U_\infty x}} = \frac{5.0}{\sqrt{\text{Re}_x}}, \quad (80)$$

where the subscript  $x$  indicates the local Reynolds number (as opposed to the global or average Reynolds number  $U_\infty L/\nu$  with  $L$  giving the overall length of the plate). The 5.0 here, in place of 3.6 with the thermal

diffusion boundary layer, is larger because of the convective transfer of low-velocity fluid away from the plate and into the free stream.

This analysis is valid as long as flow in the boundary layer remains laminar, which it remains up to a Reynolds number of  $10^5$ , beyond which the boundary layer is transitional, and past  $10^7$ , it is fully turbulent. A turbulent boundary layer behaves very differently, the turbulent equivalent to equation 80 looks like:

$$\frac{\delta_u}{x} = \frac{0.37}{\text{Re}_x^{0.2}}. \quad (81)$$

This gives boundary layer growth not as the square root of  $x$ , but as  $x^{0.8}$ , which is closer to linear.

### 3.2.2 Boundary layer and entrance length

The boundary layer shape itself allows us to determine the entrance length in confined flows, such as flow through a channel or tube. We can think of the channel as a pair of plates from which boundary layers grow and meet in the center, we consider their meeting point to define the entrance length. For a channel with plates separated by a distance  $H$ , this gives us  $\delta = H/2$  where  $x = L_e$ , the entrance length, which we can substitute into equation 79

$$\frac{H}{2} = 5.0 \sqrt{\frac{\nu L_e}{U_\infty}}, \quad (82)$$

and solve for  $L_e$

$$L_e = \frac{U_\infty H^2}{100\nu}. \quad (83)$$

Another way to look at this is to take the ratio  $L_e/U_\infty$  as the time required for a fluid particle to reach the entrance length. This timescale is proportional to the ratio of height squared to diffusivity, as with any other diffusion lengthscale. The factor of  $10^{-2}$  here is interesting: if this were a two-sided diffusion problem, the tails would reach the middle at  $H/2$  in time  $H^2/64\nu$  (recall equation 31). But due to convective momentum transfer inward, the fully-developed profile is reached even sooner.

### 3.2.3 Drag force in laminar flow past a plate

The drag force is the total force in the direction of the flow, in this case the  $x$ -direction, given by the integral of that component of traction over the surface, as discussed in section 1.6.3. In this case, with  $\hat{y}$  as the plate normal, the  $x$ -component of the traction is equal to

$$t_x = -\tau_{xy} = \mu \left( \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right), \quad (84)$$

and since  $u_y = 0$  for all  $x$  on the plate, the second partial derivative vanishes. To calculate  $\partial u_x/\partial y$ , we need the derivative of the Blasius solution  $d(u_x/U_\infty)/d\beta$ , which at  $\beta = 0$  is 0.332. Therefore, the shear strain rate at  $y = 0$  is:

$$\frac{\partial u_x}{\partial y} = U_\infty \frac{d(u_x/U_\infty)}{d\beta} \frac{\partial \beta}{\partial y} = 0.332 U_\infty \sqrt{\frac{U_\infty}{\nu x}}, \quad (85)$$

and the traction given by:

$$t_x = \mu \frac{\partial u_x}{\partial y} = 0.332 \sqrt{\frac{\rho \mu U_\infty^3}{x}}. \quad (86)$$

Note the apparent singularity at  $x = 0$ , the leading edge of the plate. Here of course the assumptions in the above analysis do not hold, in particular  $x \gg \delta_u$ , so this singularity is not quite physical; nonetheless, the singularity has a finite integral which has decent accuracy.

To calculate the drag force, we integrate this over the surface of the plate from  $x = 0$  to  $L$  and across the width  $W$ , which gives:

$$F_d = W \int_{x=0}^L 0.332 \sqrt{\frac{\rho \mu U_\infty^3}{x}} dx = 0.664 W \sqrt{\rho \mu U_\infty^3 L}. \quad (87)$$

### 3.3 Friction factor

As was discussed in section 2.3.1, the dynamic pressure is given by  $\frac{1}{2}\rho U^2$ , where  $U$  is the characteristic velocity of the system. The product of the dynamic pressure and area gives an estimate of the drag force, an estimate which improves with increasing Reynolds number. We thus define the dimensionless *friction factor*  $f$  as the constant of proportionality between the actual drag force and this estimate of dynamic pressure times area:

$$F_d = f \cdot \frac{1}{2}\rho U^2 \cdot A. \quad (88)$$

In Reynolds number régimes where the friction factor is roughly constant, we say that constant is the *drag coefficient*.

Recall that for flow past a sphere, the low Reynolds number drag force was proportional to the viscosity, velocity, and lengthscale, due to the dominance of the viscous shear in drag. At high Reynolds number, the convective terms dominate momentum transfer, so the inertia of the fluid becomes more important, and the drag force is proportional to density, velocity squared, and area. Because low-Reynolds number flow is less sensitive to details of geometry, such as surface roughness, we define the *drag coefficient* for the high-Reynolds number case, the wisdom of which will be seen more clearly in the tube flow and flow past a sphere applications below.

#### 3.3.1 Friction factor on a plate

On a plate, two types of friction factors are used: a local friction factor  $f_x$  which relates dynamic pressure to shear traction

$$t_x = f_x \frac{1}{2}\rho U^2, \quad (89)$$

and a global or average coefficient  $f_L$  which relates dynamic pressure and plate area to drag force, defined in equation 88. In both cases, the characteristic velocity for flow past a plate is  $U_\infty$ .

To calculate the local friction factor, we take the ratio of  $t_x$  from equation 86 to the dynamic pressure:

$$f_x = \frac{t_x}{P_d} = \frac{0.332\sqrt{\frac{\rho\mu U_\infty^3}{x}}}{\frac{1}{2}\rho U_\infty^2} = 0.664\sqrt{\frac{\mu}{\rho U_\infty x}} = \frac{0.664}{\sqrt{\text{Re}_x}}. \quad (90)$$

Interestingly, the Reynolds number emerges on the right side. This is generally the case: the local and global friction factors will be functions of the Reynolds number, and in many other cases, the geometry details, and in compressible flow the Mach number.

The global/average friction factor is the ratio of  $F_d$  from equation 87 to dynamic pressure and total area  $LW$ :

$$f_L = \frac{F_d}{P_d \cdot A} = \frac{0.664W\sqrt{\rho\mu U_\infty^3 L}}{\frac{1}{2}\rho U_\infty^2 \cdot LW} = 1.328\sqrt{\frac{\mu}{\rho U_\infty L}} = \frac{1.328}{\sqrt{\text{Re}_L}}. \quad (91)$$

Again the friction factor is related to the Reynolds number.

For turbulent boundary layers, local drag traction is chaotic, making  $f_x$  meaningless except as a time average. But  $f_L$  is easier to define and measure, though different sources give different expressions. In Bird, Stewart and Lightfoot's *Transport Phenomena*, the relationship given is:

$$f_L = \frac{0.455}{(\log \text{Re}_L)^{2.58}}, \quad (92)$$

whereas Poirier and Geiger (cited earlier) give the relationship:

$$f_L = \frac{0.146}{\text{Re}_L^{0.2}}. \quad (93)$$

Either way, the friction factor in the turbulent régime is higher than for laminar flow, due to the increased efficiency of momentum transfer from the fluid to the plate by convection in turbulent vortices.

### 3.3.2 Flow in a tube

For fully-developed laminar pressure-driven flow in a tube (discussed in section 2.2.4), velocity is given by

$$u_z = \frac{\Delta P}{4\mu L} (R^2 - r^2), \quad (94)$$

with average velocity

$$u_{z,av} = \frac{\Delta P}{8\mu L} R^2. \quad (95)$$

The velocity profile can be written in terms of its average, with the resulting expression independent of what drives the flow:

$$u_z = 2u_{z,av} \left(1 - \frac{r^2}{R^2}\right). \quad (96)$$

The traction on the inner wall at  $R = r$  is then:

$$t_z = -\tau_{rz} = \mu \left( \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \right)_{r=R} = \frac{4\mu u_{z,av}}{R}. \quad (97)$$

The Reynolds number uses diameter as the characteristic lengthscale and average velocity as characteristic velocity, so we can express  $t_z$  in terms of diameter  $d$  and divide by dynamic pressure to give the friction factor:

$$f = \frac{8\mu u_{z,av}/d}{\frac{1}{2}\rho u_{z,av}^2} = \frac{16\mu}{\rho u_{z,av} d} = \frac{16}{\text{Re}}. \quad (98)$$

Note that in fully-developed flow where this is valid, the traction is uniform along the length of the tube, so total drag force is the product of traction and area, and the local and average friction factors are equal.

In turbulent flow, friction factor depends on Reynolds number and the dimensionless roughness given by the ratio of roughness feature size to radius, as shown in figure 5. (This will of course also be a function of the shape of the roughness, but this ratio can be defined to give correct answers for various roughness geometries.) This illustrates the wisdom of using the dynamic pressure and area  $\frac{1}{2}\rho U^2 A$  to define the drag coefficient, and not, say,  $\mu UL$ , since the former gives constant drag coefficient for a given roughness, whereas the latter would give multiple lines.

### 3.3.3 Flow past a sphere

When written, this section will discuss the friction factor curve for flow past a sphere, similar to figure 5.

### 3.3.4 Flow past a cylinder

This section will refer to figure 2, and come up with another curve like figure 5.

## 3.4 Exercises

1. Based on equation 38, show that the friction factor for Stokes flow past a sphere is equal to

$$f = \frac{24}{\text{Re}}, \quad (99)$$

where Reynolds number uses sphere diameter as the characteristic lengthscale.

2. Drag force on a flat plate

A flat plate measuring  $1\text{m} \times \frac{1}{4}\text{m}$  is held in the air such that two edges (the “sides”) are parallel to the wind, and two (the leading and trailing edges) are perpendicular to it.

Data: air  $\rho \simeq 1.9 \frac{\text{kg}}{\text{m}^3}$ ,  $\mu \simeq 10^{-5} \frac{\text{N}\cdot\text{s}}{\text{m}^2}$ .

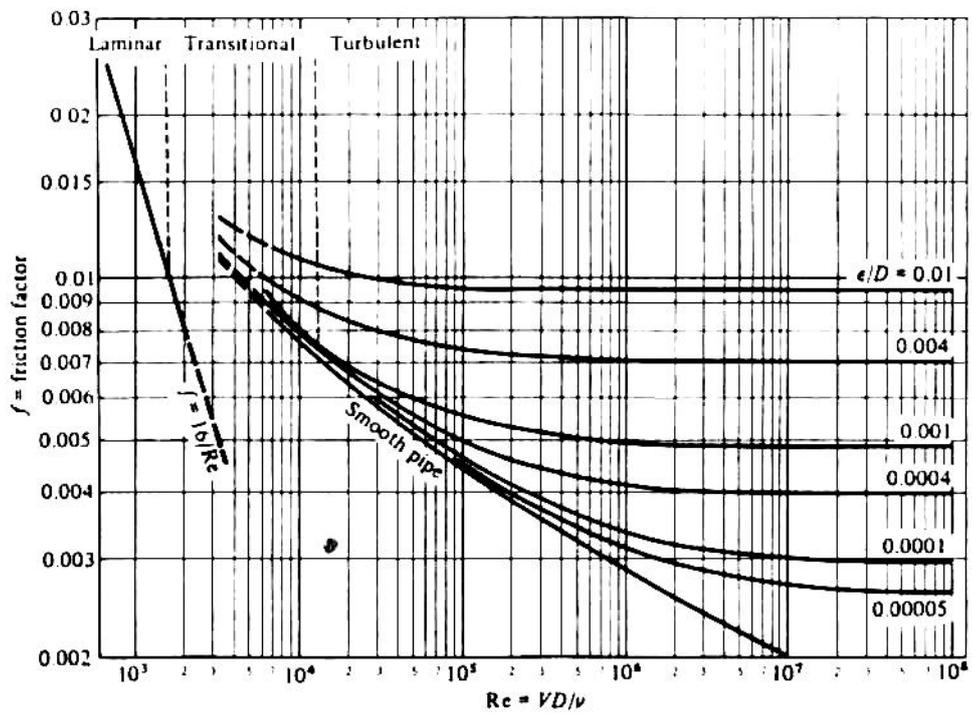


Figure 5: Friction factor for flow in a tube, from Poirier and Geiger, p. 80.

- (a) Based on what you know about drag on a flat plate, is the drag higher when the longer (1m) edges are parallel to the wind, or perpendicular to it?
- (b) For the lower-drag case in part 2a, calculate the total drag force on the plate (*i.e.* on both sides of it) for a (very low) wind velocity of  $1 \frac{\text{cm}}{\text{sec}}$ .
- (c) Calculate the drag force for the same plate in a  $10 \frac{\text{m}}{\text{s}}$  wind.

### 3. Fluid flow in blood vessels

You are designing a material for use as artificial blood vessels, and would like to know the shear forces and pressures which these blood vessels must sustain. You decide to treat blood vessels as cylindrical tubes with constant flow rate to calculate those quantities for vessels of various sizes, from the aorta to capillaries.

For a very rough estimate needed for initial design studies, you may use the properties of water:  $\rho = 1000 \frac{\text{kg}}{\text{m}^3}$ ,  $\mu = 10^{-3} \frac{\text{kg}}{\text{m}\cdot\text{s}}$ .

- (a) Use an approximate volume of a heart ventricle and pulse to calculate the average flow rate through the aorta.
- (b) Based on this flow rate, and aorta diameter of 0.7 cm, calculate the average velocity and Reynolds number of flow in the aorta. Is flow laminar or turbulent?
- (c) Use this Reynolds number and curve in figure 5 to calculate the friction factor and approximate shear traction on the walls of the aorta.
- (d) Suppose a capillary is 1mm long and  $50\mu\text{m}$  in diameter. If the average flow velocity is  $1 \frac{\text{mm}}{\text{sec}}$ , what is the pressure drop from one end to the other?
- (e) Give at least two important reasons why the above analysis breaks down for blood flow in real arteries and capillaries.

## 4 Estimating Heat and Mass Transfer Coefficients

Heat and mass transfer coefficients are proportionality constants which relate the heat or solute diffusion flux at the surface of a solid to the difference between the temperature or concentration at the solid surface and that in the bulk of the fluid, away from the boundary layer which forms in a transport-limited situation. They are defined as:

$$h \equiv \frac{\vec{q} \cdot \hat{n}}{T_s - T_\infty}, \quad (100)$$

$$h_D \equiv \frac{\vec{J} \cdot \hat{n}}{C_s - C_{eq}}, \quad (101)$$

where  $C_{eq}$  is the concentration in the solid in equilibrium with the bulk fluid concentration. These coefficients describe the ability of the solute or heat to diffuse through the boundary layer.

For the heat transfer coefficient, the heat transfer coefficient is approximately given by the ratio of fluid conductivity to thermal boundary layer thickness:

$$h \sim \frac{k_{fl}}{\delta_T}, \quad (102)$$

when this is substituted into equation 100, we recover something like equation 6. The mass transfer coefficient is a bit more complicated, due the equilibrium relationship between concentrations in the solid and fluid, but we could write equation 102 for the mass transfer coefficient by multiplying by a partition coefficient appropriate to the local chemistry.

## 4.1 Convective heat and mass transfer

Convective heat and mass transfer occur due to motion of a fluid which carries local properties with it, as was discussed in sections 2.1.1 and 3.1. As discussed earlier, the convective flux of solutes and heat used to derive the transport equations are given by  $\vec{u}C$  and  $\vec{u}\rho c_p T$  respectively; these give rise to the  $\vec{u} \cdot \nabla$  terms of the substantial derivative, but also for compressible fluids, a  $\nabla \cdot \vec{u}$  term, providing accumulation due to net influx of fluid. The complete, general equations for solute and heat transfer are thus given by:

$$\frac{DC}{Dt} + C\nabla \cdot \vec{u} = \nabla \cdot (D\nabla C) + G, \quad (103)$$

$$\rho c_p \left( \frac{DT}{Dt} + T\nabla \cdot \vec{u} \right) = \nabla \cdot (k\nabla T) + \dot{q}. \quad (104)$$

As with the momentum boundary layer discussed in section 3, the solute concentration or temperature is often very uniform throughout most of the fluid, and affected by a solid in a thin layer near a solid, called a concentration or thermal boundary layer.

For solutes and momentum, we generally speak of two types of convective transfer: *forced convection*, in which an external force drives a fluid past a solid, and *natural convection*, in which the difference in temperature or solute concentration near the solid vs. the rest of the fluid results in density differences, and the resulting buoyancy force due to gravity drives the flow. Both of these will be discussed presently, but first a definition or two.

### 4.1.1 Relative diffusivities: the Prandtl number

The *Prandtl number* gives the ratio of momentum diffusivity to diffusivity of another quantity, such as solutes or heat. We thus can have multiple Prandtl numbers for various phenomena, and in a multicomponent solution with varying diffusivities for the various solutes, even multiple mass transfer Prandtl numbers. For solutes and heat, the Prandtl numbers are defined as:

$$\text{Pr} = \frac{\nu}{D}, \quad (105)$$

$$\text{Pr} = \frac{\nu}{\alpha}. \quad (106)$$

The mass transfer Prandtl number  $\nu/D$  is also called the *Schmidt number*.

Because it is a ratio of properties, the Prandtl number itself is a property of the fluid. Mass transfer Prandtl numbers in liquids are uniformly very large (much greater than one), because solute diffusion happens much more slowly than viscous momentum transfer. Heat transfer Prandtl numbers are not quite so large, and for liquid metals in particular, can be substantially smaller than one.

### 4.1.2 Dimensionless transfer coefficients: the Nusselt number

The Nusselt number is defined as the ratio of the heat or mass transfer coefficient to thermal conductivity or diffusivity of the fluid, scaled by a lengthscale:

$$\text{Nu} \equiv \frac{hL}{k_{fl}}, \quad (107)$$

$$\text{Nu} \equiv \frac{h_D L}{D_{fl}} \text{ or } \frac{k''L}{D_{fl}}, \quad (108)$$

where  $k''$  in the second can be a heterogeneous reaction rate coefficient instead of a mass transfer coefficient. Interestingly, equation 102 allows us to rewrite the Nusselt number as the ratio of the lengthscale to the boundary layer thickness, for example in heat conduction:

$$\text{Nu} \sim \frac{(k_{fl}/\delta_T)L}{k_{fl}} = \frac{L}{\delta_T}. \quad (109)$$

A large Nusselt number therefore corresponds to a small boundary layer, and vice versa. The mass transfer Nusselt number is sometimes called the *Sherwood number*, and like the mass transfer coefficient, its physical interpretation is complicated by a partition coefficient.

This physical interpretation illustrates the difference between the Nusselt number and the formally-similar Biot number, which is defined using the thermal conductivity or diffusivity of a solid:

$$\text{Bi} \equiv \frac{hL}{k_s} = \frac{L/k_s}{1/h}, \quad (110)$$

$$\text{Bi} \equiv \frac{h_D L}{D_s} = \frac{L/D_s}{1/h}. \quad (111)$$

The lengthscale here is the thickness of a solid, and the Biot number represents the ratio of resistance to diffusion or conduction through the solid to diffusion or conduction through the boundary layer in the fluid. Again, the relevant diffusion and convection are happening in two different phases; in the Nusselt number they're both happening together in the fluid. The mass transfer Biot number is sometimes called the *Damköhler number*.

## 4.2 Forced convection past a plate

The case of forced convection past a plate is analogous to both sections 3.1 and 3.2, in that we have both thermal and momentum boundary layers—and they have different thicknesses. There are two limiting cases here: a thicker thermal boundary layer, and a thicker momentum boundary layer, which correspond to low and high Prandtl numbers respectively, and behave quite differently.

All of the analyses below are given in terms of heat transfer, but apply equally well to mass transfer, with the caveat that if the definition of the mass transfer coefficient  $h_D$  in equation 100 is used, then the result must be multiplied by a partition coefficient.

### 4.2.1 Low Prandtl number case: moving solid revisited

If the Prandtl number is low (less than about 0.05), typically with liquid metals, then the fluid diffuses heat much faster than momentum, and the thermal boundary layer will be much thicker than the momentum boundary layer. Because of this, the thermal boundary layer is mostly moving at a uniform velocity, and the analysis in section 3.1 approximately applies: heat transfer is governed by equation 69, which when  $\delta_t \ll x$  simplifies to equation 70, whose solution is in equation 71, and the thermal boundary layer thickness is approximately given by equation 73.

Assuming the temperature distribution does not affect the viscosity of the fluid, the thermal and momentum equations are *weakly-coupled*, that is, the temperature distribution depends on the velocity distribution but not vice versa. In this case, velocity distribution is given by the Blasius solution, discussed above in section 3.2.1. The two boundary layer thicknesses, for the temperature and velocity, are thus given by equations 73 and 79, and their ratio is:

$$\frac{\delta_u}{\delta_T} = \frac{3.6\sqrt{\frac{\alpha x}{U_\infty}}}{5.0\sqrt{\frac{\nu x}{U_\infty}}} = 0.72\sqrt{\frac{\nu}{\alpha}} = 0.72\text{Pr}^{0.5}. \quad (112)$$

We can next estimate the heat transfer coefficient by calculating the normal flux  $q_y$  at the wall where  $y = 0$ :

$$q_y = -k\frac{\partial T}{\partial y} = -k\frac{\partial}{\partial y} \left[ (T_\infty - T_s)\text{erf} \left( \frac{y}{2\sqrt{\alpha x/U_\infty}} \right) \right]. \quad (113)$$

The derivative of the erf at zero is  $2/\sqrt{\pi}$ , and defining  $h$  according to  $T_s - T_\infty$  (equation 100) lets us change the sign:

$$q_y = k(T_s - T_\infty)\frac{2}{\sqrt{\pi}}\frac{1}{2\sqrt{\alpha x/U_\infty}} = h(T_s - T_\infty), \quad (114)$$

and allows us to solve for  $h$ :

$$h = \frac{k}{\sqrt{\pi}} \sqrt{\frac{U_\infty}{\alpha x}}. \quad (115)$$

Defining the local Nusselt number  $\text{Nu}_x$  as the Nusselt number with distance from the plate leading edge as the lengthscale (just as we defined the local Reynolds number in section 3.2.1, equation 80) quickly gives us that dimensionless heat transfer coefficient in terms of two other dimensionless parameters:

$$\text{Nu}_x = \frac{hx}{k} = \frac{1}{\sqrt{\pi}} \sqrt{\frac{U_\infty x \nu}{\nu \alpha}} = \frac{1}{\sqrt{\pi}} (\text{Re}_x \text{Pr})^{0.5}, \quad (116)$$

where  $\text{Re}_x \text{Pr}$  is used in place of  $\text{Pe}_x$  to show contrast with the expression which will be given in section 4.2.2. When the Prandtl number is not so small but “intermediate”, we can use the following correction to equation 116:

$$\text{Nu}_x = \frac{0.564 \text{Re}_x^{1/2} \text{Pr}^{1/2}}{1 + 0.90 \sqrt{\text{Pr}}}. \quad (117)$$

Note that this gives a slightly lower Nusselt number, and therefore a slightly smaller heat transfer coefficient than equation 116. This is because the former equation was derived using uniform fluid velocity equal to  $U_\infty$ , whereas the fluid in the momentum boundary layer is not moving that fast, and convection in that important region is weaker. When the thermal boundary layer is smaller than the momentum boundary layer, corresponding to a high Prandtl number, the Nusselt number will be smaller still.

#### 4.2.2 High Prandtl number case

In the high-Prandtl number case, the heat and mass transfer behavior are dominated by the region in which velocity is nearly linear in  $y$ . Convection is considerably weaker than in the low-Prandtl number case, resulting in a thicker thermal/solute boundary layer and lower heat transfer coefficient. Rigorous analysis of this case is beyond the scope of this lecture series, so the expressions for relative boundary layer thicknesses and Nusselt number are given without derivation:

$$\frac{\delta_u}{\delta_T} = 0.975 \text{Pr}^{1/3}, \quad (118)$$

$$\text{Nu}_x = 0.332 \text{Re}_x^{1/2} \text{Pr}^{0.343}. \quad (119)$$

These expressions are reasonably accurate for Prandtl numbers above about 0.6.

#### 4.2.3 Effects of turbulence

When flow is turbulent, the boundary layer grows much more quickly than the laminar  $\delta_u \sim x^{0.5}$ , as discussed in sections 3.2.1 and 3.3.1. Extended to combined flow and heat transfer, the thermal boundary layer will be also be thicker than in the laminar case, and the heat transfer coefficient larger. It is worth noting that the eddies in turbulent flow transfer heat and solutes about as well as momentum. Thus if we define the turbulent viscosity  $\mu_t$ , thermal conductivity  $k_t$  and solute diffusivity  $D_t$  as their apparent values in a turbulent flow field, then the *turbulent Prandtl number*  $\nu_t/\alpha_t$  or  $\nu_t/D_t$  is very close to one. For this reason, in turbulent flow past a plate, the velocity, thermal, and solutal boundary layers have approximately the same thicknesses, as given by equation 81.

### 4.3 Natural convection

Natural convection is quite different from forced convection, in that the dominant flow driver is not an externally-imposed velocity, but gravitational buoyancy due to temperature or solute concentration variations in the fluid. Typically, a solid is at a very different temperature or surface equilibrium concentration from the surrounding fluid, and the heat or solute will diffuse into the fluid, giving the fluid in the region near the solid a different density from the fluid away from the solid. If this region is denser, then fluid there will sink; if less dense, then it will rise.

The simplest natural convection case involves heat or mass transfer from a vertical plate into a semi-infinite fluid. (I think I’ll leave this section unfinished for now.)

#### 4.4 Summary: solution procedure

The analyses above for forced and natural convection were based on a very simple planar geometry. But solving problems for different geometries from those above is not difficult: if the geometry is relatively simple, then there will be similar relationships between the Nusselt number and the Reynolds and Prandtl numbers. This is analogous to the friction factor, which is given as different functions of the Reynolds number for a flat plate, tube, and sphere above (sections 3.3.1, 3.3.2 and 3.3.3).

In all of these cases, the procedure for estimating the local or global heat or mass transfer coefficient is pretty nearly uniform:

1. Calculate the Prandtl number from fluid properties.
2. Based on the flow characteristics or temperatures, determine the Reynolds number (forced convection) or Grashof number (natural convection).
3. Using the Prandtl and Reynolds/Grashof numbers, determine the flow régime ( $\delta_u \leq x/\delta_u \gg x$ , laminar/turbulent, low/high Prandtl number) and choose the Nusselt number expression corresponding to this régime for the geometry in question.
4. Calculate the Nusselt number using this expression.
5. Solve equation 107 or 108 for  $h$  or  $h_D$  and calculate it.

That's all there is to it.

#### 4.5 Exercises

1. Based on equations 117 and 119, develop expressions for the global/average Nusselt number  $Nu_L$  under low-Prandtl number and high-Prandtl number conditions, analogous to the global/average friction factor  $f_L$  (e.g. in equation 91). (Assume flow in the boundary layer is laminar.)

## A Summary of Dimensionless Numbers

Heat Transfer	Mass Transfer
<p>Biot Number</p> <p>Ratio of conductive to convective H.T. resistance</p> <p>Determines uniformity of temperature in solid</p> $Bi = hL/k_{solid}$	<p>M.T. Biot Number</p> <p>Ratio of diffusive to reactive or conv MT resistance</p> <p>Determines uniformity of concentration in solid</p> $Bi = kL/D_{solid} \text{ or } h_D L/D_{solid}$
<p>Fourier Number</p> <p>Ratio of current time to time to reach steady-state</p> <p>Dimensionless time in temperature curves, used in explicit finite difference stability criterion</p> $Fo = \alpha t/L^2$	$Fo = Dt/L^2$
<p>Knudsen Number:</p> <p>Ratio of gas molecule mean free path to process lengthscale</p> <p>Indicates validity of line-of-sight (<math>&gt; 1</math>) or continuum (<math>&lt; 0.01</math>) gas models</p> $Kn = \frac{\lambda}{L} = \frac{kT}{\sqrt{2}\pi\sigma^2 PL}$	
<p>Reynolds Number:</p> <p>Ratio of convective to viscous momentum transport</p> <p>Determines transition to turbulence, dynamic pressure vs. viscous drag</p> $Re = \rho UL/\mu = UL/\nu$	
<p>Prandtl Number</p> <p>Ratio of momentum to species diffusivity</p> <p>Determines ratio of fluid/HT BL thickness</p> $Pr = \nu/\alpha = \mu c_p/k$	<p>M.T. Prandtl (Schmidt) Number</p> <p>Ratio of momentum to thermal diffusivity</p> <p>Determines ratio of fluid/MT BL thickness</p> $Pr = \nu/D = \mu/\rho D$
<p>Nusselt Number</p> <p>Ratio of lengthscale to thermal BL thickness</p> <p>Used to calculate heat transfer coefficient <math>h</math></p> $Bi = hL/k_{fluid}$	<p>M.T. Nusselt (Sherwood) Number</p> <p>Ratio of lengthscale to diffusion BL thickness</p> <p>Used to calculate mass transfer coefficient <math>h_D</math></p> $Bi = h_D L/D_{fluid}$
<p>Grashof Number</p> <p>Ratio of natural convection buoyancy force to viscous force</p> <p>Controls the ratio of lengthscale to natural convection boundary layer thickness</p> $Gr = g\beta\Delta TL^3/\nu^2$	$Gr = g\beta_C\Delta CL^3/\nu^2$
<p>Peclet Number</p> <p>Ratio of convective to diffusive heat/mass transport in a fluid</p> <p>Used to determine plug flow/perfect mixing (CSTR) continuous flow model validity</p> $Pe = RePr = UL/\alpha$	$Pe = RePr = UL/D$
<p>Rayleigh Number</p> <p>Ratio of natural convective to diffusive heat/mass transport</p> <p>Determines the transition to turbulence</p> $Ra = GrPr = g\beta\Delta TL^3/\nu\alpha$	$Ra = GrPr = g\beta_C\Delta TL^3/\nu D$