3.185 – Recitation Notes

November 6/7, 2003

Topics covered

- Two approaches to solve a fluid dynamic problem:
    1. *Momentum balance*
    2. *Navier-Stokes' equation*

Momentum Balance

The momentum balance approach is simply writing a force balance for a volume element. Note that in fluid dynamics, force can be manifested as the usual applied pressure over an area but also in terms of the rate of momentum in/out $\dot{m}v$ due to fluid motion (convection). Here are some frequently encountered "force" terms: *(Suppose velocity gradient is in the x direction and the fluid is moving in the z direction)*

Shear force on surface *x* (momentum transport due to viscosity)

$$\Delta z \Delta y \tau_{xz}\big|_x$$

Shear force on surface $x + \Delta x$ (momentum transport due to viscosity)

$$\Delta z \Delta y \tau_{xz}\big|_{x+\Delta x}$$

Rate of momentum in across surface at *z* (momentum transport due to convection)

$$v_z \underbrace{\left(\Delta x \Delta y \rho v_z\right)\big|_z}_{mass\ flux}$$

Rate of momentum out across surface at $z + \Delta z$ (momentum transport due to convection)

$$v_z \underbrace{\left(\Delta x \Delta y \rho v_z\right)\big|_{z+\Delta z}}_{mass\ flux}$$

Gravity force

$$\Delta x \Delta y \Delta z \rho g_z$$

Pressure force acting on surface at *z*

$$\Delta x \Delta y P_z\big|_z$$

Pressure force acting on surface at $z + \Delta z$

$$\Delta x \Delta y P_z\big|_{z+\Delta z}$$

The momentum balance equation is given by equating the sum of all forces in z-direction (fluid flow direction) to zero, $\sum F_z = 0$.

Dividing through by the volume ($\Delta x\ \Delta y\ \Delta z$) gives a differential equation that can be used to find the velocity and shear stress profile.

Navier Stokes' Equation

Continuity equation

$$\nabla \cdot v = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

Momentum equation (constant ρ and η)

$$\rho\left[\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right] = -\frac{\partial P}{\partial x} + \eta\left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right] + \rho g_x$$

$$\rho\left[\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right] = -\frac{\partial P}{\partial y} + \eta\left[\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right] + \rho g_y$$

$$\rho\left[\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right] = -\frac{\partial P}{\partial z} + \eta\left[\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z$$

This can be rewritten as:

$$\rho\frac{Dv}{Dt} = -\nabla P + \eta\nabla^2 v + \rho g$$

Remember from last recitation, we discussed that the substantial derivative is the change in the particle's frame of reference. In this case, the change in velocity with time can be considered as the acceleration felt by a particle. By Newton's law of motion, acceleration in a body is caused by a non-zero net force, which is given by the product of mass and acceleration. The left hand side of the momentum equation is a product of mass (per unit volume) and acceleration; therefore, it is reasonable to look at the right hand side as a sum of forces acting on the particle in a fluid. Hence the similarity in solution of the momentum balance and the Navier Stokes' approach is not surprising. In fact, the Navier Stokes' equation is derived using the momentum balance approach for the general case. Recall from our discussion during last recitation, a steady state flow, i.e. $\frac{\partial v}{\partial t} = 0$, does not require the velocity to be constant with time from a particle's frame of reference as long as there is a non-zero net force acting on the particle.