<u>3.185 - Recitation Notes</u>

October 30/31, 2003

Topics Covered

- Mass balance in motion
- Substantial derivative and it's meaning

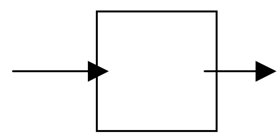
Simple mass conservation equation for 1-D

Mass Accumulation = Mass In – Mass out

Note:

1. Don't worry about diffusion
2. Do worry about mass flow by motion such as fluid flow

Mass accumulation is pretty obvious



$$accum. = \Delta x \Delta y \Delta z \frac{\partial \rho}{\partial t}$$

Here we assume density can fluctuate

Mass in:

Since we ignore any diffusion, *total mass in* would only be the mass flow rate (density * velocity) across the boundary

$$mass_{in} = \rho_x u_x \Delta y \Delta z$$

Similarly, *total mass out* is

$$mass_{out} = \rho_{x+\Delta x} u_x \Delta y \Delta z \ ,$$

but note

1. we assume velocity in the x direction is constant (for simplicity)
2. the density is different at x and at $x + \Delta x$

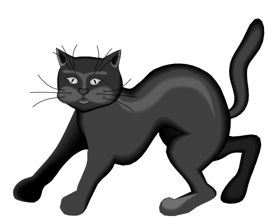
Putting all together

$$\Delta x \Delta y \Delta z \frac{\partial \rho}{\partial t} = \rho_x u_x \Delta y \Delta z - \rho_{x+\Delta x} u_x \Delta y \Delta z$$

$$\frac{\partial \rho}{\partial t} = \frac{\rho_x u_x - \rho_{x+\Delta x} u_x}{\Delta x}$$

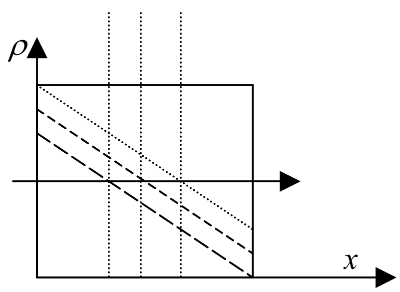
$$\frac{\partial \rho}{\partial t} = -u_x \frac{\partial \rho}{\partial x}$$

$$\frac{\partial \rho}{\partial t} + u_x \frac{\partial \rho}{\partial x} = \frac{D\rho}{Dt} = 0$$



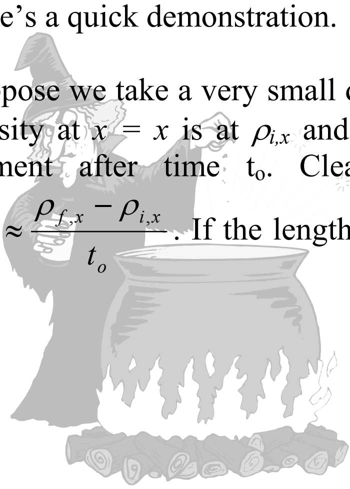
The last equation is rather familiar, the left hand side of the equation is the substantial derivative (of course only with the x component). It is interesting to point out that for mass conservation in a volume, the substantial derivative of density *($D\rho/Dt$)* is zero even though we allow the local density to fluctuate with time.

Suppose density is constant, it is required that either $u_x = 0$ or $\frac{\partial \rho}{\partial x} = 0$ in order to satisfy the conservation equation $\frac{\partial \rho}{\partial t} + u_x \frac{\partial \rho}{\partial x} = 0$. In the first case, the system is not a moving body/fluid system because there is no fluid/body velocity. The frame of reference is the same for both the laboratory observer and the particle. In the second case, the laboratory observer sees no change in density with time because we specified that $\frac{\partial \rho}{\partial t} = 0$. The particle also sees no change in density as it moves around because $\frac{\partial \rho}{\partial x} = 0$. Suppose we still maintain the conservation of mass, but instead the density is a function of time, i.e. $\frac{\partial \rho}{\partial t} \neq 0$. For the laboratory observer, the density is changing with time. In order to satisfy the conservation of mass and since there is no diffusion, the particle and the density gradient $\frac{\partial \rho}{\partial x}$ must be in a certain configuration. For example, let the density inside the



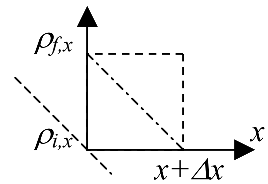
differential element increases with time, then the product $u_x \frac{\partial \rho}{\partial x}$ must be negative. For fluid/body moving in a positive x-direction, the density gradient must be negative. Following the trajectory of the particle with time (denoted by the vertical lines), it is shown here that the density remains the same from the particle's frame of reference. Is this just a coincident?

Here's a quick demonstration.



Suppose we take a very small differential element. As the particle enters the element, the density at $x = x$ is at $\rho_{i,x}$ and $\rho_{f,x}$ is the density at $x = x$ when the particle leaves the element after time t_o. Clearly, the change in density with time is give by $\frac{\partial \rho}{\partial t} \approx \frac{\rho_{f,x} - \rho_{i,x}}{t_o}$. If the length of the differential element is Δx, then the velocity of the

particle is $u_x = \frac{\Delta x}{t_o}$. According to the conservation equation derived earlier,



$$\frac{\partial \rho}{\partial t} = -u_x \frac{\partial \rho}{\partial x}$$

$$\frac{\partial \rho}{\partial x} = -\frac{1}{u_x}\frac{\partial \rho}{\partial t}$$

$$\frac{\partial \rho}{\partial x} = -\frac{t_o}{\Delta x}\frac{\rho_{f,x} - \rho_{i,x}}{t_o} = \frac{\rho_{i,x} - \rho_{f,x}}{\Delta x}$$

At time $t = t_o$, we know that the density at $x = x$ is $\rho_{f,x}$, and we want to know the density at $x = x+\Delta x$. The slope of the density profile can also be written as $\frac{\partial \rho}{\partial x} \approx \frac{\rho_{f,x+\Delta x} - \rho_{f,x}}{\Delta x}$. Comparing the last two equations, it is evident that $\rho_{f,x+\Delta x} = \rho_{i,x}$; therefore, the density is the same from the particle's frame of reference. The above demonstration assumes a linear density profile; however, it is conceptually straightforward that any curves can be approximated as linear when the differential element is sufficiently small. Hence, this should be a rather general result and not a coincident. One last note…

Since the substantial derivative of density is zero by conservation equation and we have shown that the change in density with time is also zero from the particle's frame of reference, it is therefore reasonable to think of the substantial derivative as the change in density with time from the particle's point of view.

Now imagine the "stuff" for which the conservation of mass has been derived are actually lots of tiny carriers capable of carrying such quantities like energy, momentum or money. If we assume the flow of these quantities is only possible through the tiny carriers, then we can write a conservation equation that has such general forms:

Total accumulation = $mass_{in}$ * *amt carried per carrier* – $mass_{out}$ * *amt carried per carrier*

This is a general conservation equation for moving body/fluid without any diffusive component or body force. In the case of energy, the amount carried per carrier is given by the enthalpy per mass and multiplying this with $mass_{in}/mass_{out}$ gives the flux of enthalpy into/out of the differential volume due to the migration of these tiny carriers.

The same is true for momentum. Since momentum is the product of mass and velocity, the flux of momentum into/out of the differential element is given by the product of $mass_{in}/mass_{out}$ and velocity, u. Note that momentum flux has the same dimension as force. Hence, the balance of momentum flux is also a force balance. It is important to remember both definitions when you are writing the differential equation for a fluid system. A typical force balance for a fluid element takes into account the following contributions:

- Rate of momentum in/out across surfaces on the side by momentum diffusion
- Rate of momentum in/out across cross-sectional surface by convective transport
- Pressure force acting on the cross-sectional surface
- Body force such as gravity

Happy Halloween!