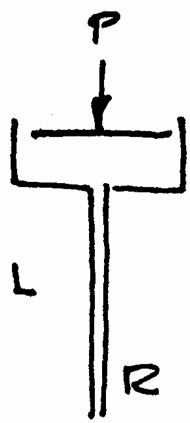
Rheometry - $\tau = \eta \dot{\gamma}$

- Capillary rheometer

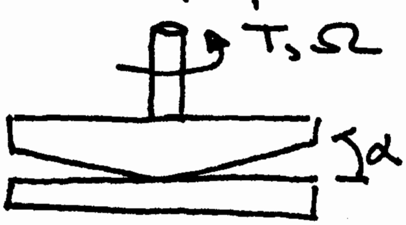


$$0 = -\frac{\partial P}{\partial z} + \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)$$

$$u_z = \frac{R^2}{4\mu}\cdot\frac{\Delta P}{L}\left[1-\left(\frac{r}{R}\right)^2\right]$$

$$Q = \int_0^R u_z(r)\cdot 2\pi r\,dr = \frac{\pi R^4}{8\mu}\cdot\frac{\Delta P}{L}$$

- Cone & plate



$$\eta = \frac{3\alpha T}{2\pi R^2 \Omega}$$

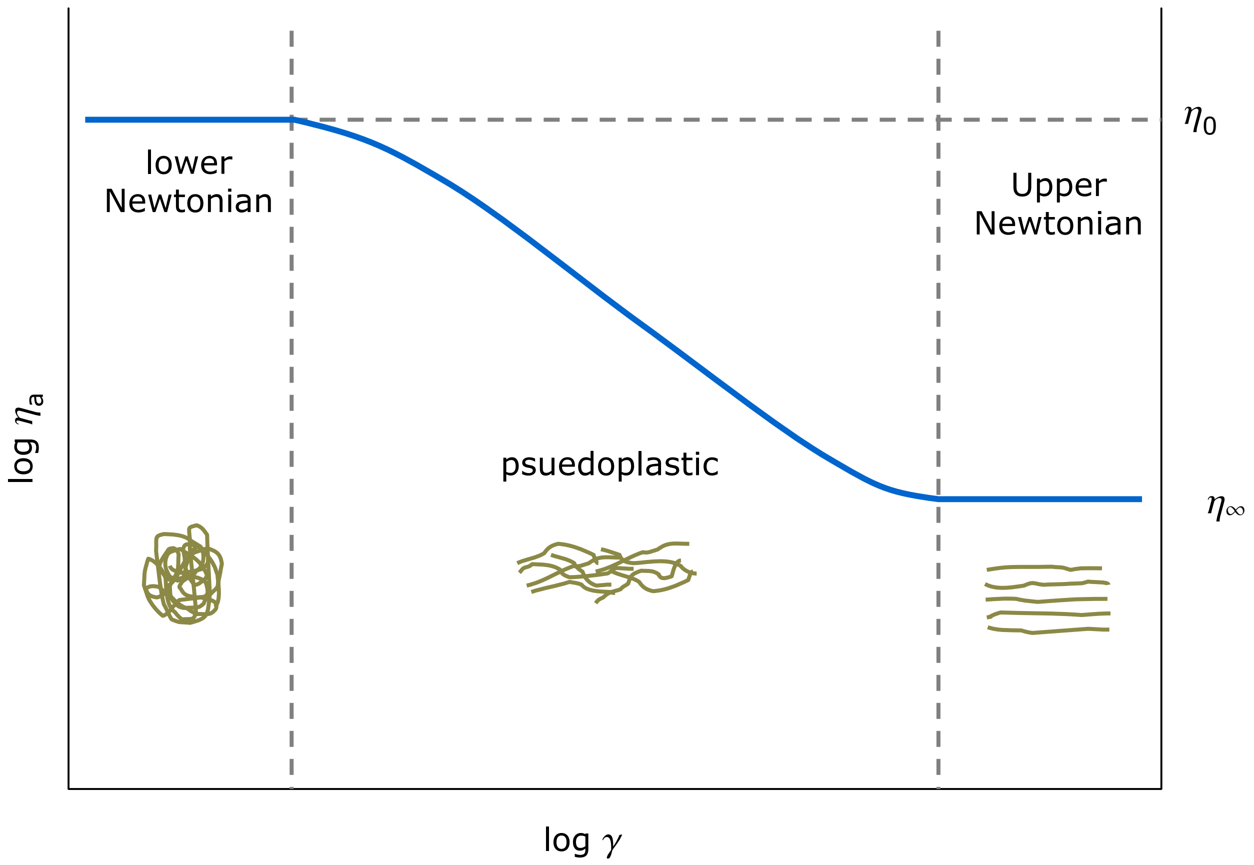
$$\dot{\gamma} = \Omega/\alpha$$

- Oscillatory rheometer

$$\gamma = \gamma_0 \cos \omega t\,, \quad \tau = \tau_0 \cos(\omega t + \delta)$$

$$\eta^* = \frac{\tau}{\dot{\gamma}} = \frac{(\tau_0' + i\tau_0'')e^{i\omega t}}{\gamma_0 (i\omega) e^{i\omega t}} = \frac{\tau_0''}{\gamma_0 \omega} - i\,\frac{\tau_0'}{\gamma_0 \omega}$$

$$\eta' = \frac{G''}{\omega}\,, \quad \eta'' = \frac{G'}{\omega}$$



η_0
lower
Newtonian
Upper
Newtonian
log η_a
psuedoplastic
η_∞
log γ

Viscoelastic Fluids

Maxwell model:

$$\dot{\gamma} = \dot{\gamma}_s + \dot{\gamma}_d = \frac{\dot{\tau}}{k} + \frac{\tau}{\eta}$$

$$\tau_{ij} + \lambda \frac{\partial \tau_{ij}}{\partial t} = \eta \Delta_{ij} \qquad \left\{ \begin{array}{l} \lambda = \eta / k \\ \\ \Delta_{ij} = u_{i,j} + u_{j,i} \end{array} \right.$$

Oldroyd codeformational derivative:

$$\frac{\mathcal{D}}{\mathcal{D}t}\tau_{ij} = \frac{\partial \tau_{ij}}{\partial t} + u_k \frac{\partial \tau_{ij}}{\partial x_k} - \tau_{kj}\frac{\partial u_i}{\partial x_k} - \tau_{ik}\frac{\partial u_j}{\partial x_k}$$

White - Metzner model:

$$\tau_{ij} + \lambda \frac{\mathcal{D}\tau_{ij}}{\mathcal{D}t} = \eta \Delta_{ij}$$

Simple shear flow $(u = \dot{\gamma} y,\ v = w = 0)$

$$\begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix} - \lambda\dot{\gamma} \begin{bmatrix} 2\tau_{12} & \tau_{22} & \tau_{23} \\ \tau_{22} & 0 & 0 \\ \tau_{32} & 0 & 0 \end{bmatrix} = \eta\dot{\gamma} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\tau_{22} = 0$$
$$\tau_{12} = \eta\dot{\gamma}$$
$$\tau_{11} = -\lambda\dot{\gamma}(2\tau_{12}) = 2\lambda\eta\dot{\gamma}^2$$

$$\Psi_{12} \equiv \frac{\tau_{11} - \tau_{22}}{\dot{\gamma}^2} = 2\lambda\eta$$

FINITE ELEMENT EQUATIONS

- Differential equation:

$$0 = Q + k\nabla^2 T$$

- Interpolation among nodal unknowns:

$$\tilde{T}(x,y) = N_j(x,y)T_j$$

- Galerkin weighted residual:

$$\int_V N_i(Q + k\nabla^2\tilde{T})\,dV = \mathcal{R} \approx 0$$

- Substituting, integrating by parts:

$$k_{ij}T_j = q_i$$

where

$$k_{ij} = \int_V \nabla N_i k \nabla N_j\,dV + \oint_\Gamma N_i h N_j\,d\Gamma$$

and

$$q_i = \int_V N_i Q\,dV + \oint_\Gamma N_i h T_a\,d\Gamma$$

FEM Algorithms for Flow

- Penalty method for incompressibility

$$p = \alpha\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \longrightarrow K_p = \int (m^T B_i)^T \alpha (m^T B_j)\, dV$$

$$m^T = (1, 1, 0)$$

- Streamfunction

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}$$

$$\left[\int_V \nabla N_i \nabla N_j \, dV\right] \psi_j = \int_V N_i \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) dV$$

- Streamline upwinding

$$\rho c\, u \nabla T \longrightarrow \int_V N_i \rho c u \nabla N_j \, dV$$

- Time stepping

$$c\left(\frac{da}{dt}\right) + Ka = f$$

$$c\left(\frac{a_{n+1} - a_n}{dt}\right) + K\left[\theta a_{n+1} + (1-\theta) a_n\right] = f$$

$$\left[\frac{c}{\Delta t} + \theta K\right] \Delta a = f - K a_n$$