Lecture 2, Structure. 3.054

Structure of cellular solids

2D honeycombs:
- Polygonal cells pack to fill 2D plane — Fig.2.3a
- Prismatic in 3^{rd} direction

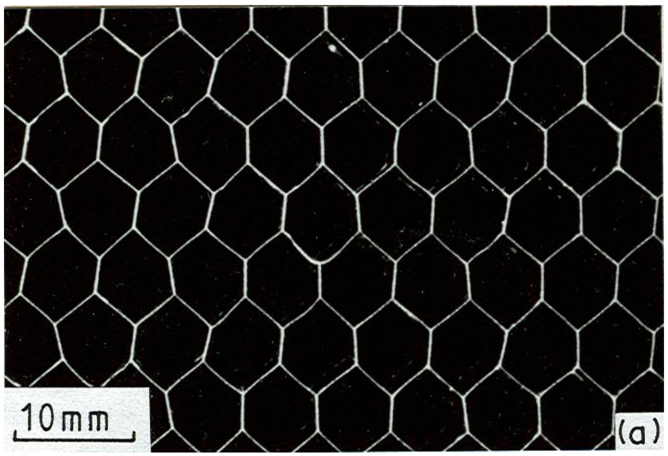
3D foams:
- Polyhedral cells pack to fill space — Fig.2.5

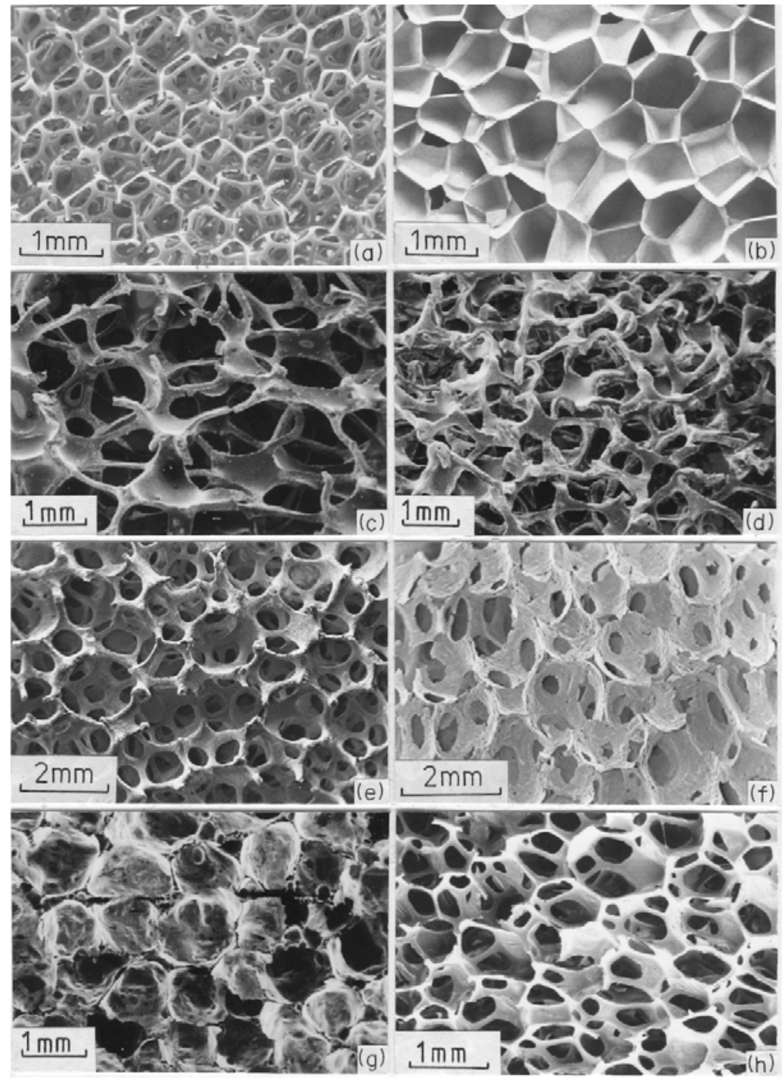
Properties of cellular solid depend on:

- Properties of solid it is made from (ρ_s, E_s, σ_{ys}...)
- Relative density, ρ^*/ρ_s (= volume fraction solids)
- Cell geometry
- Cell shape $\rightarrow$ anisotropy
- Foams - open vs. closed cells

  open: Solid in edges only; voids continuous

  closed: Faces also solid; cells closed off from one another
- Cell size - typically not important





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Relative Density

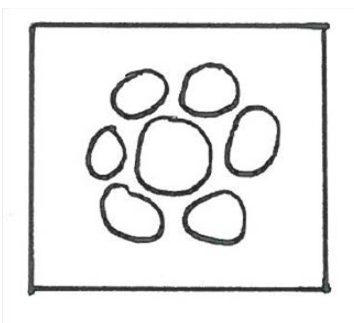
$\rho^* =$ density of cellular solid
$\rho_s =$ density of solid it made from

$$\frac{\rho^*}{\rho_s} = \frac{M_s}{V_T}\ \frac{V_s}{M_s} = \frac{V_s}{V_T} = \text{ volume fraction of solid } (= 1-\text{porosity})$$

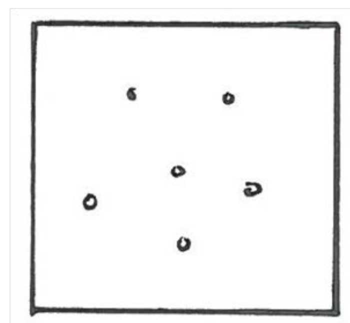
Typical values:

collagen - GAG scaffolds:	$\rho_*/\rho_s = 0.005$
typical polymer foams:	$0.02 < \rho_*/\rho_s < 0.2$
soft woods:	$0.15 < \rho_*/\rho_s < 0.4$

- As ρ^*/ρ_s increases, cell edges (and faces) thicken, pore volume decreases
- In limit $\rightarrow$ isolated pores in solid



$\rho^*/\rho_s < 0.3$
cellular solid



$\rho^*/\rho_s > 0.8$
isolated pores in solid

Unit Cells

2D honeycombs: - Triangles, squares, hexagons Fig.2.11
- Can be stacked in more than one way
- Different number of edges/vertex
- Fig. 2.11 (a)-(e) isotropic; others anisotropic

3D foams: Rhombic dodecahedra and tetrakaidecahedra pack to fill space Fig.2.13
(apart from triangles, squares, hexagons and prisms)

[Greek: hedron = face; do = 2; deca = 10; tetra = 4; kai = and]

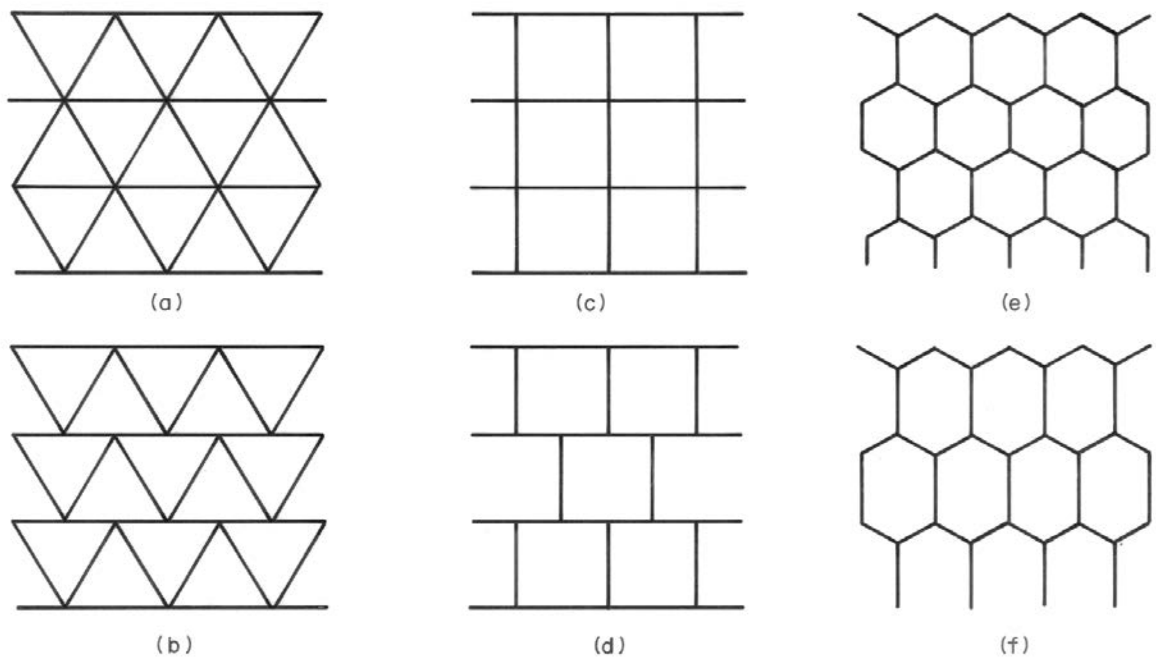
Tetrakaidecahedra - bcc packing; geometries in Table 2.1

- Foams often made by blowing gas into a liquid
- If surface tension is only controlling factor and if it is isotropic, then the structure is one that minimizes surface area at constant volume

Kelvin (1887): tetrakaidecahedron with slightly curved faces is the single unit cell that packs to fill space plus minimizes surface area/volume Fig.2.4

Weaire-Phelan (1994): identified “cell” made up of 8 polyhedra that has slightly lower surface area/volume (obtained using numerical technique - “surface evolver”)

Unit Cells: Honeycombs



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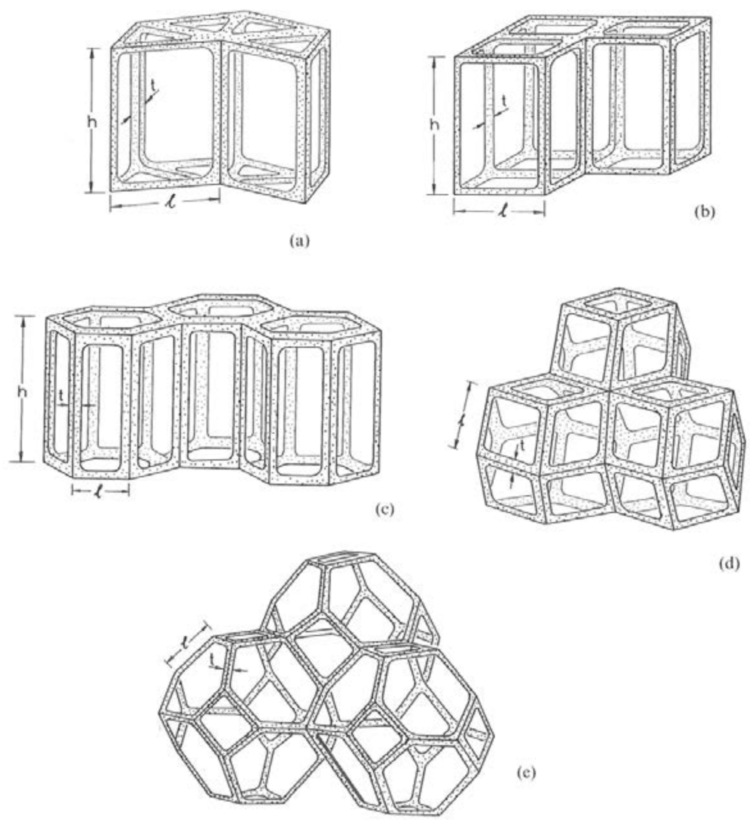
Unit Cells: Foams



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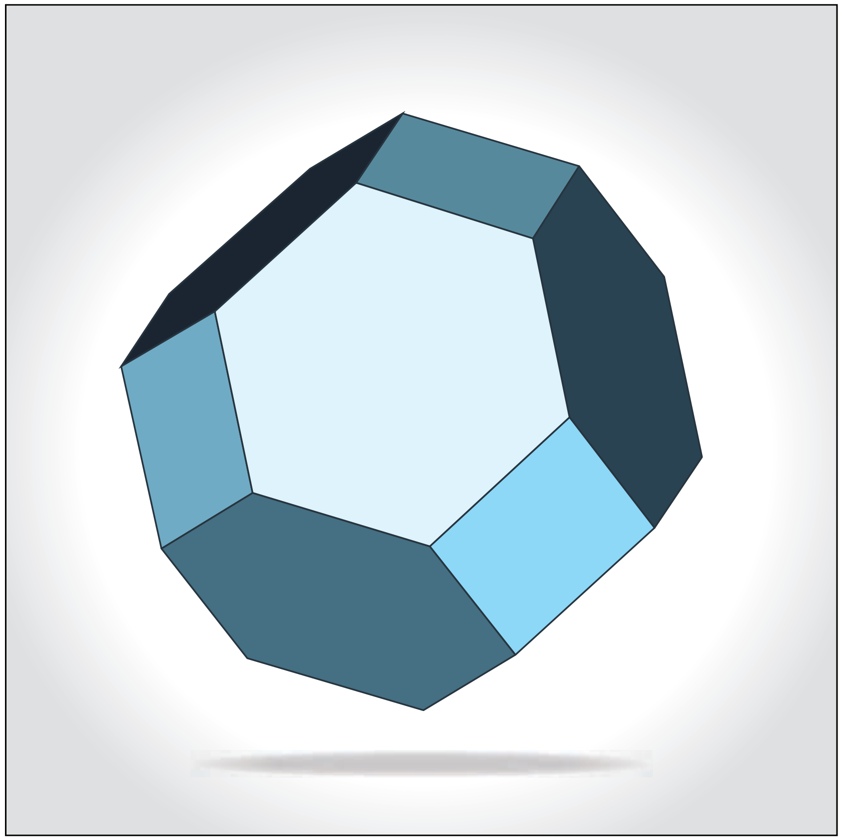
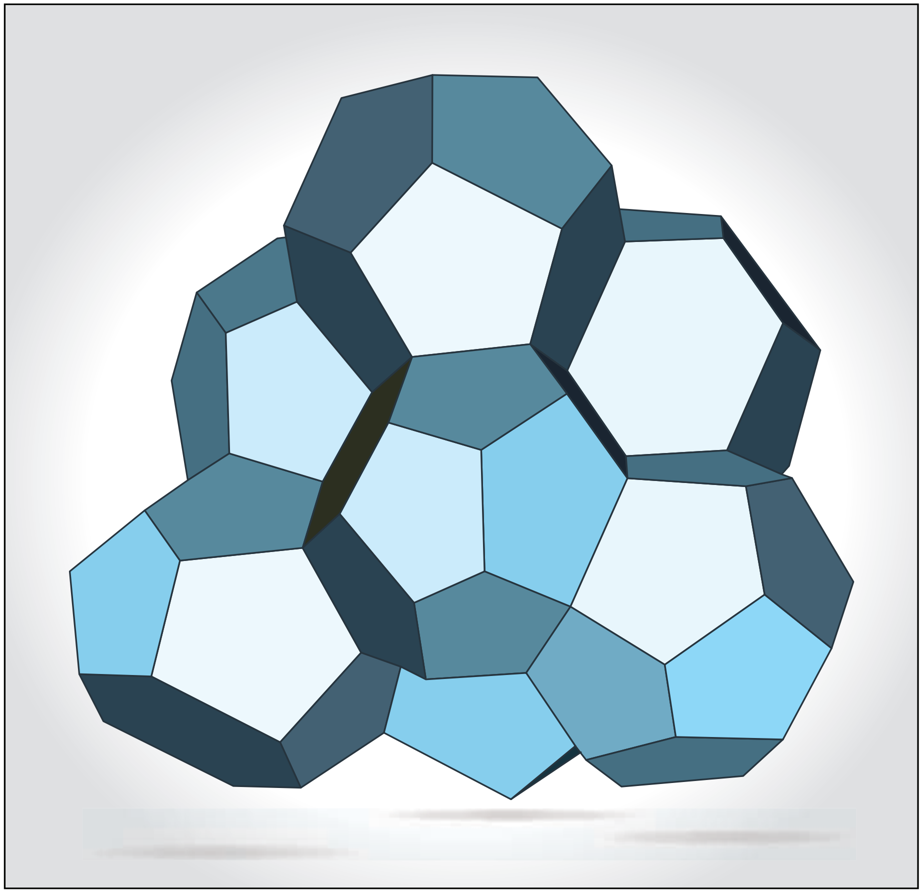
Unit Cells: Kelvin Tetrakaidecahedron



Kelvin's tetrakaidecahedral cell.
Source: Professor Denis Weaire; Figure 2.4 in Gibson, L. J., and M. F. Ashby. *Cellular Solids Structure and Properties*. Cambridge University Press, 1997.

Unit Cells: Weaire-Phelan

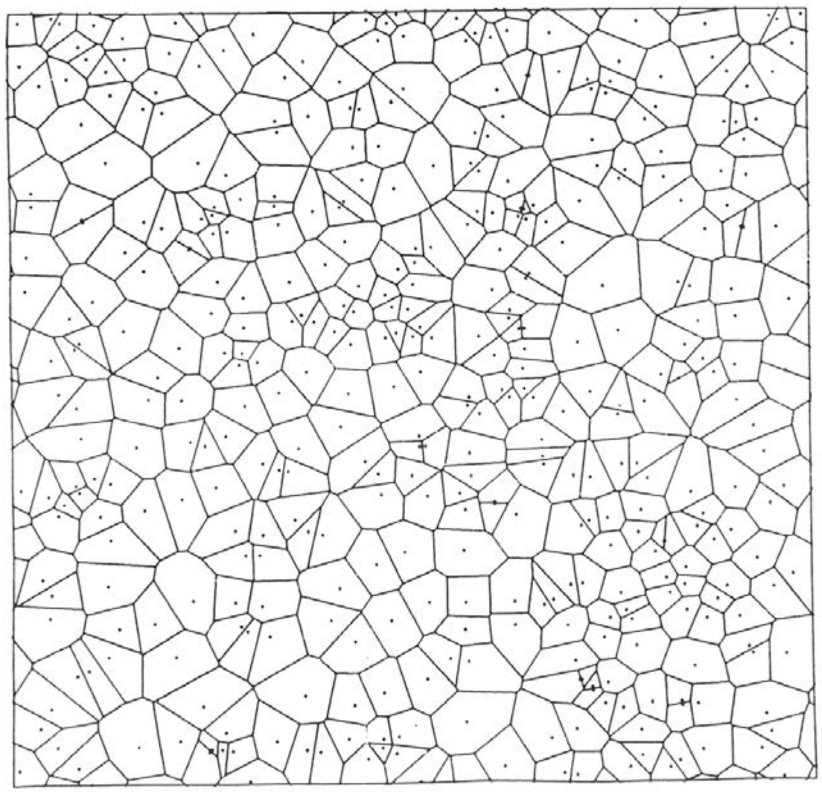


Weaire and Phelan's unit cell.
Source: Professor Denis Weaire; Figure 2.4 in Gibson, L. J., and M. F. Ashby. *Cellular Solids: Structure and Properties*. Cambridge University Press, 1997.

Voroni Honeycombs and Foams

- Foams sometimes made by supersaturating liquid with a gas and then reducing the pressure, so that bubbles nucleate and grow
- Initially form spheres; as they grow, they intersect and form polyhedral cells
- Consider an idealized case: bubbles all nucleate randomly in space at the same time and grow at the same linear rate
  - obtain Voroni foam (2D Voroni honeycomb)
  - Voroni structures represent structures that result from nucleation and growth of bubbles Fig.2.14a
- Voroni honeycomb is constructed by forming perpendicular bisectors between random nucleation points and forming the envelope of surfaces that surround each point
- Each cell contains all points that are closer to its nucleation point than any other
- Cells appear angular
- If specify exclusion distance (nucleation points no closer than exclusion distance) then cells less angular and of more similar size Fig.2.14b

Voronoi Honeycomb



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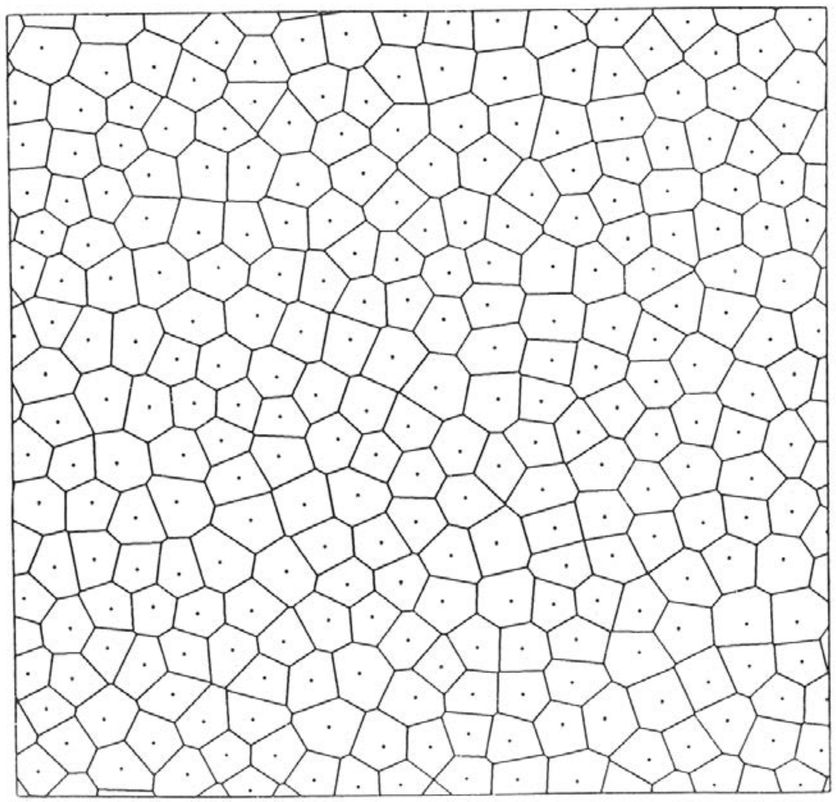
Voronoi Honeycomb with Exclusion Distance



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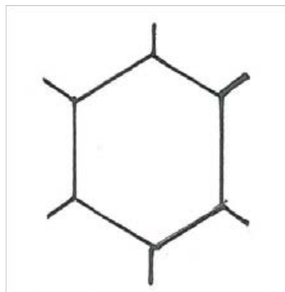
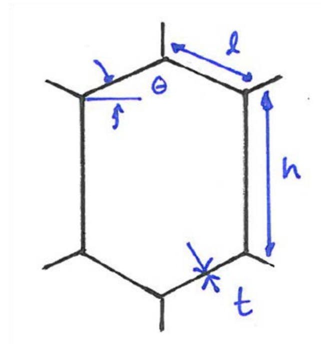
Cell Shape, Mean Intercept Length, Anisotropy

Honeycombs



regular hexagon:
isotropic in plane



elongated hexagon: anisotropic
h/l, θ define cell shape

Foams

- Characterize cell shape, orientation by mean intercept lengths
- Consider circular test area of plane section
- Draw equidistant parallel lines at $\theta = 0°$
- Count number of intercepts of cell wall with lines:

Huber paper Fig.9

N_c = number of cells per unit length of line

$L(\theta = 0°) = \frac{1.5}{N_c}$

Mean Intercept Length

Figures removed due to copyright restrictions. See Fig. 9: Huber, A. T., and L. J. Gibson. "Anisotropy of Foams." *Journal of Materials Science* 23 (1988): 3031-40.

Mean intercept

- Increment θ by some amount (eq. 5°) and repeat
- Plot polar diagram of mean intercept lengths as $f(\theta)$
- Fit ellipse to points (in $3D$, ellipsoid)
- Principal axes of ellipsoid are principal dimensions of cell
- Orientation of ellipse corresponds to orientation of cell
- Equation of ellipsoid: $Ax_1^2 + Bx_2^2 + Cx_3^2 + 2Dx_1x_2 + 2Ex_1x_3 + 2Fx_2x_3 = 1$
- Write as matrix M: $M = \begin{bmatrix} A & B & E \\ D & B & F \\ E & F & C \end{bmatrix}$
- Can also represent as tensor "fabric tensor"
- If all non-diagonal elements of the matrix are zero, then diagonal elements correspond to principal cell dimensions

Connectivity

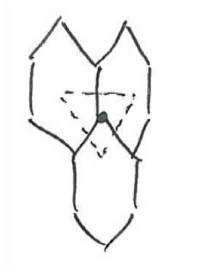
- *Vertices* connected by *edges* which surround *faces* which enclose *cells*
- Edge connectivity, Z_e = number of edges meeting at a vertex
  typically $Z_e = 3$ for honeycombs
  $Z_e = 4$ for foams
- Face connectivity, Z_f = number of faces meeting at an edge
  typically, $Z_f = 3$ for foams

Euler's Law

- Total number of vertices, V, edges, E, faces, F, and cells, C is related by Euler's Law (for a large aggregate of cells):

  $$2D: \quad F - E + V = 1$$
  $$3D: \quad -C + F - E + V = 1$$

For an irregular, 3-connected honeycomb (with cells with different number of edges), what is the average number of sides/face, $\bar{n}$?



$Z_e = 3 \quad \therefore E/V = 3/2$ (each edge shared between 2 vertices)
If F_n = number of faces with n sides, then:
$\sum \frac{nF_n}{2} = E$ (factor of 2 since each edge separated two faces)

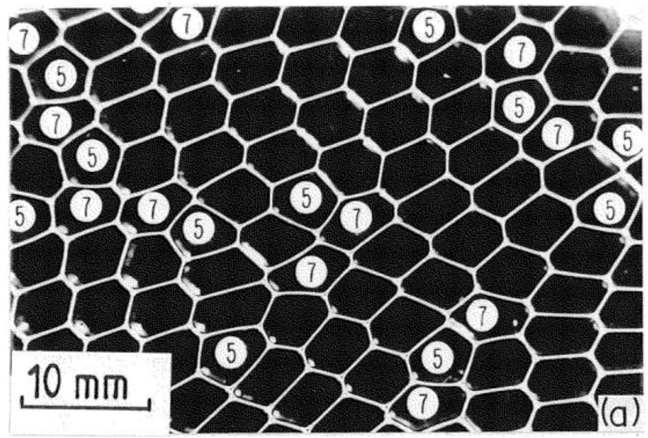
Using Euler's Law:

$$F - E + \frac{2}{3}E = 1$$
$$F - \frac{1}{3}\sum \frac{nF_n}{2} = 1$$
$$6F - \sum nF_n = 6$$
$$6 - \frac{\sum nF_n}{F} = \frac{6}{F}$$

As F becomes large, RHS $\to 0$
$\frac{\sum nF_n}{F} =$ average number of sides per face, $\bar{n}$
$\bar{n} = 6$
For 3-connected honeycomb, average number of sides *always* 6.

Fig.2.9a

Euler's Law



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Soap Honeycomb

Aboav-Weaire Law

- Euler's Law: for 3-connected honeycomb, average number of sides/face=6
- Introduction of a 5-sided cell requires introduction of 7-sided cell, etc
- Generally, cells with more sides (in $2D$) (or faces, in $3D$) than average, have neighbors with fewer sides (in $2D$) (or faces, in $3D$) than average
- **Aboav** - observation in $2D$ soap froth

  Weaire - derivation
- $2D$: If a candidate cell has n sides, then the average number of sides of its n neighbors is $\bar{m}$:

$$\bar{m} = 5 + \frac{6}{n} \qquad (2D)$$

Lewis' Rule

- Lewis examined biological cells and 2D cell patterns
- Found that area of a cell varied linearly with the number of its sides

$$\frac{A(n)}{A(\bar{n})} = \frac{n - n_0}{\bar{n} - n_0}$$

$A(n)$ = area of cell with n sides
$A(\bar{n})$ = area of cell with average number of sides, $\bar{n}$
n_0 = constant (Lewis found $n_0 = 2$)

- Holds for Voronoi honeycomb; Lewis found holds for most of other $2D$ cells
- Also, in $3D$:

$$\frac{V(f)}{V(\bar{f})} = \frac{f - f_0}{\bar{f} - f_0}$$

$V(f)$ = volume of cell with f faces
$V(\bar{f})$ = volume of cell with average number of faces, $\bar{f}$
f_0 = constant, ≈ 3

Modeling cellular solids - structural analysis

Three main approaches:

1. Unit cell
   - E.g. honeycomb-hexagonal cells
   - Foam - tetrakaidecahedra (but cells not all tetrakaidecahedra)
2. Dimensional analysis
   Foams - complex geometry, difficult to model exactly
   - instead, model mechanisms of deformation and failure (do not attempt to model exact cell geometry)
3. Finite element analysis
   - Can apply to random structures (e.g., 3D Voronoi) or to micro-computed tomography information.
   - Most useful to look at local effects (e.g., defects - missing struts - osteoporosis size effects)